Please read the following instructions carefully:

- There are **eight problems** in this exam.
- There is **one bonus** problem.
- You have **90 minutes** to complete the exam
- Please write each solution on a separate page.
- You **must have your camera on** during the exam.
- This is a **closed book, closed notes exam**. You must not consult any resource while attempting the exam.
- Upload your work to Gradescope.
- Submitting the exam implies you abide by the honor pledge stated below:

I pledge on my honor that I have not given or received any unauthorized assistance on this quiz/examination

1. (10 points) A hemisphere-like shaped solid is generated by rotating the region above the x-axis, to the right of the y-axis and below the graph of the function $y = 4 - x^2$ around the y-axis. Compute the volume of the solid.

2. (10 points) The base of a solid, D, is a triangle in the xy plane with vertices $(0, 0), (3, 0), (0, 3)$. The cross sections perpendicular to the x-axis are semicricles. Write down the integral for the volume of the solid, D. **Do NOT evaluate the integral.**

3. (10 points) The parametric equation for the portion of the asteroid in the first quadrant is given by the parametric equations,

$$x(t) = R\cos^3 t, \qquad y(t) = R\sin^3 t,$$

for t between 0 and $\pi/2$. Find the length of the portion of the asteroid.

4. (10 points) A hemispherical swimming pool has a radius of 6 feet. It is completely filled with water. Set up, **but DO NOT EVALUATE**, the integral for the work required to pump all of the water to a platform 5 foot above the top of the pool. **Place the origin at the TOP of the tank**. Assume the weight of water is 62.5 pounds per cubic foot.

5. (a) (5 points) Find the inverse function of the function $y = x - 2/x$ for $x > 0$.
   (b) (5 points) Consider the function $f(x) = x^5 + x^3 + 1$. It can be shown that the function f has an inverse function. Noting that $f(1) = 3$, compute the derivative of the inverse function, $f^{-1}(y)$, at $y = 3$.

6. Solve the following questions:
   (a) (5 points) Find the derivative of the function,

$$f(x) = \log_5(e^x) + \sin^{-1}(x^2).$$

   (b) (5 points) Evaluate the integral,

$$\int 2x2^{x^2}\, dx$$

7. (10 points) Evaluate the integral,

$$\int \frac{dx}{x^2 + 3x + 3}.$$

8. Evaluate the following limits:
   (a) (5 points)

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x^2}$$

   (b) (5 points)

$$\lim_{x\to\infty} \frac{\ln(e^x - 1)}{\ln(x)}.$$

9. (5 points (bonus)) Suppose you have three different algorithms for solving the same problem and each algorithm takes a number of steps that is of the order of one of the following functions:

$$n\log_2 n, \quad n^{\frac{3}{2}}, \quad n(\log_2 n)^2$$

Which of these functions gives an algorithm that is most efficient in the long run? Why? Justify your answer.

Please read the following instructions carefully:

- There are **seven problems** in this exam.
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- **Here are some formulas/identities you may find useful:**

$$\int \tan(x)\,dx = -\ln|\cos x| + C, \qquad \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C \qquad 1 + \tan^2(x) = \sec^2(x)$$

1. (10 points) Compute the following integral:

$$\int x \tan^2(x)\, dx$$

2. (10 points) Compute the following integral:

$$\int e^{3x} \cos(x)\, dx$$

3. (10 points) Compute the following integral:

$$\int \tan^6(x) \sec^4(x)\, dx.$$

4. (10 points) Compute the following integral:

$$\int \frac{dx}{(4 - x^2)^{\frac{3}{2}}}$$

5. (10 points) Compute the following integral:

$$\int \frac{2x + 3}{x(x - 3)(x + 3)}\, dx.$$

6. (10 points) Write an approximation using the trapezoidal rule for the integral,

$$\int_0^4 e^{x^3}\, dx,$$

using $n = 4$ sub-intervals. **Do NOT find the final numerical answer.**

7. Determine whether the following improper integrals converge:

   (a) (5 points)

$$\int_1^\infty \frac{\sqrt{x + 1}}{x}\, dx$$

   (b) (5 points)

$$\int_1^\infty \frac{\ln x}{x^3}\, dx$$

8. (5 points (bonus)) In class we argued that integration by parts can be used to compute reduction formulas for some integrals. Can you compute the reduction formula for

$$I_n = \int e^{ax} \sin^n(x)\, dx\ ?$$

Please read the following instructions carefully:

- There are **eight problems** in this exam.
- There is **one bonus** problem.
- Solve the **bonus** problem only at the end.
- You have **90 minutes** to complete the exam
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1. (a) (5 points) Compute the following limit:

$$\lim_{n\to\infty} \frac{\sqrt{n}-2}{\sqrt{n}+2}$$

   (b) (5 points) Compute the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{2^{n+2}}{7^{n+3}}.$$

2. (10 points) Determine whether the following series converges. State any tests you use.

$$\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2+1}.$$

3. (10 points) Determine whether the series

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n3^{n+1}}$$

diverges, converges conditionally, or converges absolutely. State any tests that you use.

4. (10 points) Determine whether the following series converges. State any tests you use.

$$\sum_{n=0}^{\infty} \frac{2^n (n!)^2}{(2n)!}.$$

5. (10 points) Determine the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{n^2 x^n}{4^n(n^2+1)}$$

6. (10 points) Find the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{2\cdot 5\cdot 8\cdots(3n-1)(x-2)^n}{2\cdot 4\cdot 6\cdots 2n}$$

7. (10 points) Compute the power series of the function $f(x) = x^2\cos(5x)$ about $x = 0$.

8. (10 points) Express the following integral as a power series:

$$\int_0^1 e^{-x^2}\,dx.$$

9. (5 points (bonus)) In class, we argued that it is a non-trivial question to determine whether the Taylor series about the point $x = a$ associated to a function, $f(x)$, converges to $f(a)$. As an extreme example, consider the function:

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Show that the power series of $f(x)$ about $x = 0$ is identically zero. The function, however, is non-zero for $x \neq 0$. Hence, the Taylor series of this infinitely differentiable function does not converge to $f(x)$ for any $x \neq 0$.

Note: Yeesh! What's going on? Is Math broken? Well, not quite. The functions we've dealt with in class are quite special: they're *smooth* (infinitely differentiable) and *analytic* (admit power series converging to the function). The function above is only smooth, but not analytic. The study of calculus of *analytic* functions is the area of mathematics called complex analysis[1]. Real analysis (a.k.a calculus) is usually more messy, and deals only with smooth functions at best.

[1]This really is a misnomer. There's nothing complex about the subject. Analytic functions behave much more nicely than just smooth functions. Why? Since they admit power series representations, they're like *polynomials of infinite degree so to speak*. And we love polynomials, don't we?

Please read the following instructions carefully:

- There are **nine problems** in this exam.
- There is **one bonus** problem.
- Solve the **bonus** problem only at the end.
- You have **100 minutes** to complete the exam
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- **Here are some formulas/identities you may find useful:**

$$1 + \cos(x) = 2\cos^2(x/2)$$

1. (10 points) A solid is generated by rotating the region above the x-axis, to the right of the y-axis and between the graphs of the functions the graph of the function $y = x^3$ and $y = x^{\frac{1}{3}}$ around the y-axis. Compute the volume of the solid.

2. (10 points) The parametric equation of a curve is given by the equations,

$$x(t) = \cos(t) \quad y(t) = t + \sin(t), \quad 0 \leq t \leq 2\pi.$$

   Find the arc length of the the curve.

3. (10 points) Compute the following limits:

$$\lim_{x\to\infty} \frac{\ln(x)}{\ln(\ln(x))} \qquad\qquad \lim_{x\to 0} \frac{e^x}{1 - x - x^2}$$

4. Compute the following integrals:
   (a) (5 points)

$$\int \sin(\ln(x))\, dx$$

   (b) (5 points)

$$\int \sin^2(t)\cos^5(t)\, dt$$

5. Determine whether the following improper integrals converge:
   (a) (5 points)

$$\int_1^\infty \frac{e^x}{x^{\frac{1}{2}}}\, dx$$

   (b) (5 points)

$$\int_0^\infty \frac{e^{-x}}{1 + x^2}\, dx$$

6. (10 points) Write an approximation using the trapezoidal rule for the integral,

$$\int_0^1 x^{10}\sin(x)\, dx,$$

   using $n = 5$ sub-intervals. **Do NOT find the final numerical answer.**

7. (10 points) Determine the radius and interval of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{nx^n}{2^n(n+1)}$$

8. (10 points) Find the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{3\cdot 5\cdot 7\cdots(2n+1)(x-1)^n}{4\cdot 9\cdot 14\cdots(5n-1)}$$

9. (10 points) Express the following indefinite integral as a power series:

$$\int \frac{\sin(x)}{x}\, dx.$$

10. (5 points (bonus)) One cute application of the theory of infinite series is in the area of mathematics called **Fourier/Harmonic analysis**. Joseph Fourier asked the following question while solving the heat equation:

Can a periodic function $f(x)$ on $[-\pi, \pi]$ be expressed as $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$?

Note: the similarity with usual Taylor series in which each term is a polynomial of the form x^n. This is an extension of the same idea.

(a) Assuming a function can be expressed as a Fourier series, show that the coefficients a_0, a_n and b_n *have to be* given by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Note: This argument is in spirit exactly like the argument we presented for the Taylor series case, where we argued the coefficients of the Taylor series had to be $f^n(0)/n!$.

(b) Show that the Fourier series associated to the function $f(x) = |x|$ on $[-\pi, \pi]$ is given by the formula:

$$f(x) \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos(nx).$$

(c) Using the above Fourier series (and assuming its convergence to f), show that,

$$\sum_{n\geq 1,\text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Note: Woot! Now not only do we know the p-series with $p = 2$ is convergent, but we know a value for the sum!

(d) **For a freebie bonus point or so:** write down any interesting thoughts/questions you have about how to solve the preceding parts? Any thoughts on how to solve (a), (b)? Any thoughts on the comparison with infinite series? Or write down any interesting comments, feedback etc. about the bonus questions throughout the semester, or anything about the course. Anything really that comes to mind!

Note: I don't expect anyone to solve this problem, especially on an exam. This has been presented simply for your edification, and to convince you of the myriad of applications of calculus. In particular, Fourier/Harmonic analysis has applications in probability theory, signal processing, quantum mechanics, electromagnetism, theoretical computer science etc...