HOPF-RINOW THEOREM

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ABSTRACT. I was asked to substitute for a professor for the graduate-level course on differential geometry. I lectured on the relationship between geodesics and the metric space structure on Riemannian manifolds leading to the celebrated Hopf-Rinow Theorem. Below are the lecture notes I authored. If you notice any typos, please send corrections to junaida@umd.edu or junaid.aftab1994@gmail.com.

CONTENTS

In this note, all manifolds are assumed to be manifolds without boundary. We shall exclusively focus on the case of Riemannian manifolds without boundary. This is because the theory of minimizing curves becomes considerably more complicated in the presence of a nonempty boundary.

1. BACKGROUND

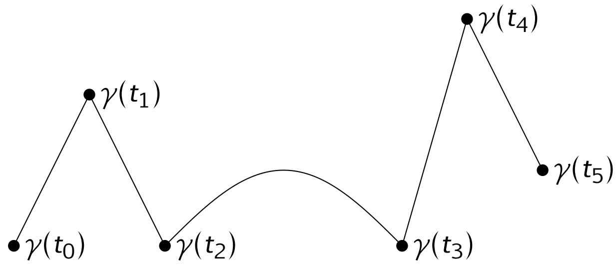
Let's recap the relevant material that has been covered in lectures thus far.

1.1. **Riemann Distance Function.** Recall that one can define the notion of length of piecewise regular curves on a Riemannian manifold, which can then be used to endow a Riemannian manifold with a metric space structure.

Definition 1.1. Let (M,g) be a Riemannian manifold. A **(parametrized) smooth curve** is a smooth map $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an interval. We say that γ is a **piecewise regular curve** (or admissible curve) if there exists a partition $(a_0,\dots,a_k)$ of I such that $\gamma'(t) \neq 0$ for all $t \in \mathrm{Int}([a_{i-1},a_i])$ for all $i = 1,\dots,k$.

The length of an admissible curve $\gamma : [a,b] \to M$ is defined as

$$L_g(\gamma) := \int_a^b |\dot\gamma(t)|_g \, dt$$



A piecewise regular curve.

Note that due to the regularity of γ, this integral is well-defined. We can now extend the most important concept from classical geometry to the setting of Riemannian manifold.

Definition 1.2. Let (M, g) be a connected Riemannian manifold. The Riemannian distance between each pair of points $p, q \in M$ is defined as

$$d_g(p,q) := \inf\{L_g(\gamma); \gamma \text{ admissible curve between } p \text{ and } q\} \tag{1}$$

Exercise 1. Let M be a connected smooth manifold. Prove that any two points in M can be joined by an admissible curve. This shows that Definition 1.2 is well-defined.

Exercise 2. Argue that the infimum in Equation (1) need not be attained. **Hint:** Consider $M = \mathbb{R}^2 \setminus \{(0,0)\}$.

Finally, we are in a position to endow the a metric space structure on a Riemannian manifold.

Proposition 1.3. *Let (M, g) be a connected Riemannian manifold. The distance function d_g is a metric on M whose metric topology induces the manifold topology.*

Proof. It is clear that $d_g(p,p) = 0, d_g(p,q) = d_g(q,p)$ and that d_g satisfies the triangle inequality. See [Lee18, Lemma 2.53, Lemma 2.54] for an elementary albeit tedious argument showing that $d_g(p,q) > 0$ if $p \neq q$. Alternatively, one can use Gauss' Lemma [Lee18, Theorem 6.9]. Moreover, see [Lee18, Theorem 2.55] for the argument that the metric topology induced by d_g is same as the manifold topology on (M, g). $\square$

1.2. **Geodesics.** Geodesics on an arbitrary Riemannian manifold are analogs of straight lines in $\mathbb{R}^n$ with the Euclidean affine connection.

Remark 1.4. *All statements below where only an affine connection is used hold for an arbitrary smooth not necessarily Riemannian manifold. However, we continue use the phrase "let (M, g) be a Riemannian manifold ..." in the hypothesis.*

Definition 1.5. Let (M, g) be a Riemannian manifold with an affine connection, ∇, and let $\gamma : I \to M$ be a smooth curve. The **acceleration** of γ is the vector field $D_t\dot{\gamma}(t)$ along γ, where D_t is the covariant derivative along γ.

Remark 1.6. *See [Lee18, Chapter 4] for more details on affine connections and covariant derivatives along smooth curves.*

Curves where the velocity $\dot{\gamma}$ is parallel along γ have a special name.

Definition 1.7. Let (M,g) be a Riemannian manifold with affine connection ∇. A smooth curve $\gamma : I \to M$ is called a **geodesic** (with respect to ∇) if its acceleration is zero, i.e., $D_t\dot{\gamma}(t) \equiv 0$.

Exercise 3. Let (M,g) be a Riemannian manifold endowed with its Levi-Civita connection. Let $\gamma : [a,b] \to M$ a smooth curve in M. Show that

$$\frac{d}{dt}\langle\dot{\gamma},\dot{\gamma}\rangle = 0$$

Hence, $|\dot{\gamma}|$ is constant. Conclude that a re-parametrization of a geodesic is again a geodesic if and only if the re-parametrization is a linear re-parametrization.

Remark 1.8. *We will call a geodesic γ on a Riemannian manifold satisfying $|\dot{\gamma}(t)| = 1$ a normal geodesic. Of course given any geodesic, the corresponding normal geodesic is nothing else but the arc-length re-parametrization of the given geodesic.*

Proposition 1.9. *Let (M,g) be a Riemannian manifold with affine connection ∇. For every $p \in M$, $w \in T_pM$, and $t_0 \in \mathbb{R}$, there exists an open interval $I \subseteq \mathbb{R}$ containing t_0 and a geodesic $\gamma : I \to M$ satisfying $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = w$. Any two such geodesics agree on their common domain.*

Exercise 4. Provide Proposition 1.9. **Hint**: Write the geodesic equation in a smooth co-ordinate charts and invoke the fundamental theorem of flows [Lee12, Chapter 9] on manifolds to argue for the existence and uniqueness of solutions to the geodesic equation. See [Chapter 3 Lee18, Chapter 4] for the proof.

Remark 1.10. *Based on the properties of the pullback connection*[1] *[Lee18, Proposition 4.38] together with the fact that being a geodesic is a local property, one can show that a local isometry maps geodesics to geodesics.*

Definition 1.11. Let (M,g) be a Riemannian manifold with an affine connection ∇. A geodesic $\gamma : I \to M$ is called **maximal** if there exists no geodesic $\widetilde{\gamma} : \widetilde{I} \to M$ with $I \subsetneq \widetilde{I}$ and $\widetilde{\gamma}|_I = \gamma$. (M,g) is **geodesically complete** if each maximal geodesic is defined on all of $\mathbb{R}$.

We now look at a geodesics of in some Riemannian manifolds.

Example 1.12. Let $M = \mathbb{R}^n$ with the Euclidean metric with the standard affine connection[2]. Since all connection coefficients are zero, the geodesic equation reads

$$\ddot{\gamma}^k(t) = 0, \quad 1 \le k \le n \qquad \gamma^k(0) = p^k, \dot{\gamma}^k(0) = w^k$$

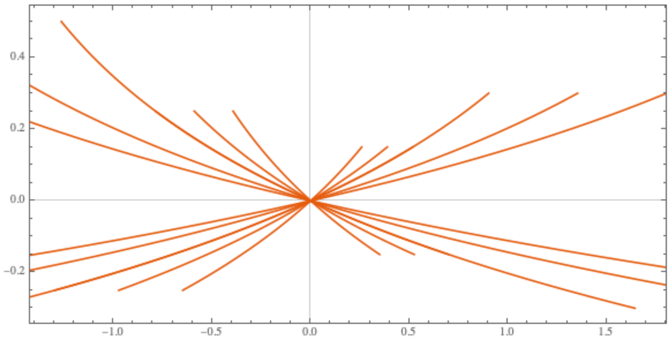
Hence, each geodesic of the form the

$$\gamma(t) = p + tw \qquad p, w \in \mathbb{R}^n$$

The same conclusion is true for Minkowski space, $\mathbb{R}^{n,1}$. In particular, $\mathbb{R}^n$ and $\mathbb{R}^{n,1}$ are geodesically complete.

[1]See [Lee18] for the definition of pullback connections.

[2]This is also the Levi-Civita connection



Geodesics with $A \neq 0$ in $\mathbb{R}^2$ with connection coefficients $\Gamma^i_{jk} = 0$ except $\Gamma^1_{12} = \Gamma^1_{21} = 1$.

Example 1.13. Endow $\mathbb{R}^2$ with an affine connection determined by the connection coefficients by $\Gamma^i_{jk} = 0$ except $\Gamma^1_{12} = \Gamma^1_{21} = 1$[3]. The geodesic equations read:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt}\frac{dy}{dt} = 0, \quad \frac{d^2y}{dt^2} = 0.$$

Consider $(x_0, y_0) \in \mathbb{R}^2$. From the second equation we have $y = At + y_0$. If $A = 0$, the solutions are

$$x = Bt + x_0, \quad y = y_0.$$

If $A \neq 0$, then

$$\frac{d^2x}{dt^2} + 2A\frac{dx}{dt} = 0 \qquad \frac{\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\frac{dx}{dt}} = -2A$$

We have $\log\left(\frac{dx}{dt}\right) = -2At + C$, so that $\frac{dx}{dt} = De^{-2At}$, where $D \neq 0$. Therefore, the equations are

$$x = \frac{D}{2A}\left(1 - e^{-2At}\right) + x_0, \quad y = At + y_0, \quad D \neq 0.$$

Some geodesics are plotted in Figure 2.

Example 1.14. Consider the cylinder

$$C := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

Note that C is surface of revolution obtained revolving the curve $\gamma(z) = (1, z)$ Therefore, a parameterization of C is given by

$$X(z, \theta) = (\cos(\theta), \sin(\theta), z) \qquad z \in \mathbb{R}\ \theta \in [0, 2\pi]$$

It can be checked that the induced metric on C is

$$dz^2 + d\theta^2$$

We consider C is endowed with its Levi-Civita connection. All Christoffel symbols/connection coefficients are zero. Thus, the geodesic equations read

$$\ddot{z} = 0 \qquad \ddot{\theta} = 0$$

[3]This is not the Levi-Civita connection.

It can be easily confirmed geodesic are vertical straightlines of the form $z = at + b, \theta = \theta_0$, helicies $z = a\theta + b$ for $a, b \neq 0$ or circles in planes of the form $z = z_0$. In particular, C is geodesically complete.

Exercise 5. Let (M, g) be a Riemannian manifold, and let ∇^0, ∇^1 be two affine connections on M. Let D be the difference tensor:

$$D(X, Y) = \nabla^0_X Y - \nabla^1_X Y$$

Prove that M has the same geodesics with respect to the ∇^0 and ∇^1 if and only if the difference tensor is an alternating tensor.

1.2.1. *Geodesics on Submanifolds.* Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Let $N \subseteq M$ be an embedded submanifold. The Levi-Civita connection on N is given by

$$(\widetilde{\nabla}_X Y)_p := \pi^T (\nabla_{\widetilde{X}} \widetilde{Y})_p \qquad p \in N \tag{2}$$

Here π^T is the orthogonal projection map from $\mathsf{T}_p M$ to $\mathsf{T}_p N$, and $\widetilde{X}, \widetilde{Y}$ denote extensions of vector fields X, Y, respectively.

Exercise 6. Prove Equation (2).

We can characterize geodesic in N computing the formula for the covariant derivative along curves in N.

Proposition 1.15. *Let (M, g) be a Riemannian manifold with an affine connection, ∇. Let $N \subseteq M$ be an embedded submanifold endowed with induced Levi-Civita connection, $\widetilde{\nabla}$, as in Equation (2). Let $\gamma : I \to M$ be a curve in M such that $\gamma(I) \subseteq N$. Let X be a vector field along γ such that $X(\gamma(t)) \in \mathsf{T}_{\gamma(t)} N$ for each $t \in I$. If $\widetilde{D}_t$ is the covariant derivative along curves in N with respect to $\widetilde{\nabla}$, then*

$$\widetilde{D}_t X = \pi^T (D_t(X))$$

Exercise 7. Prove Proposition 1.15.

Corollary 1.16. *Let $M = \mathbb{R}^n$ or $M = \mathbb{R}^{n,1}$, and let N be an embedded submanifold of M. A smooth curve $\gamma : I \to M$ is a geodesic (with respect to the induced, tangential connection on M) if and only if $\ddot{\gamma}(t)$ is orthogonal to $\mathsf{T}_{\gamma(t)} M$ for all $t \in I$.*

Proof. This follows at once from the previous proposition and the fact tha $D_t \dot{\gamma} = \ddot{\gamma}$ on $M = \mathbb{R}^n$ or $M = \mathbb{R}^{n,1}$. □

Example 1.17. (**Geodesics on $\mathbb{S}^n(R)$**) Let's invoke Corollary 1.16 to compute geodesics on $\mathbb{S}^n(R)$. Let $p \in \mathbb{S}^n(R)$, and let $v \in \mathsf{T}_p \mathbb{S}^n(R)$. We then have that $p \perp v$. Let $\hat{v} = Rv/|v|$. Consider the smooth curve

$$\gamma(t) = \cos atp + \sin(at)\hat{v}, \quad a = |v|/R$$

Clearly, $\gamma(0) = p$. Note that

$$\dot{\gamma}(t) = -a \sin(at) p + a \cos(at) \hat{v}$$
$$\ddot{\gamma}(t) = -a^2 \cos(at) p - a^2 \sin(at) \hat{v}$$

Note that $\dot{\gamma}(0) = a\hat{v} = v$. Since $\ddot{\gamma}(t)$ is parallel to $\gamma(t)$, we have that $\ddot{\gamma}(t)$ is orthogonal to $\mathsf{T}_{\gamma(t)}\mathbb{S}^n(R)$. By Corollary 1.16, $\gamma(t)$ is a geodesic such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$[4]. We label this geodesic $\gamma_{p,v}$. Each such $\gamma_{p,v}$ has period $2\pi/a$, and has constant speed. The image of $\gamma_{p,v}$ is the great circle formed by the intersection of $\mathbb{S}^n(R)$ with the linear subspace spanned by p and $\hat{v}$. Hence, all geodesics on $\mathbb{S}^n(R)$ are great circles.

1.3. **Exponential Map & Normal Coordinates.** One can study the behaviour of all geodesics at point on a Riemannian manifold. This information is conveniently encoded in the exponential map.

Remark 1.18. *Everything in this section works verbatim for semi-Riemannian manifolds and, in fact, even for any other affine connection. As before, All statements below where only an affine connection is used hold for an arbitrary smooth not necessarily Riemannian manifold.*

In what follows, let $\gamma_{p,v}$ denote a geodesic such that $\gamma_{p,v}(0) = p$ and $\dot{\gamma}_{p,v}(0) = v$. We start off with a simple lemma.

Lemma 1.19. *(**Rescaling Lemma**) Let (M, g) be a Riemannian manifold with an affine connection ∇. For every $p \in M$, $v \in T_pM$, and $c, t \in \mathbb{R}$, we have that*

$$\gamma_{p,cv}(t) = \gamma_{p,v}(ct)$$

whenever both sides are defined.

Proof. Consider the curve $\gamma_{p,v} : I \to M$. Define a new curve

$$\tilde{\gamma} : c^{-1}I \to M, \quad t \mapsto \gamma_{p,v}(ct)$$

Clearly,

$$\tilde{\gamma}(0) = \gamma(0) = p \qquad \tilde{\gamma}'(t) = c\dot{\gamma}_0(0) = cv$$

We now check that $\tilde{\gamma}$ satisfies the geodesic equation. Hence, initial point and velocity of both sides of (4.7) are equal. Working in a local coordinate chart, it is easy to check that

$$D_t\tilde{\gamma}'(t) = c^2 D_t\gamma'(ct) = 0$$

By existence and uniqueness, $\tilde{\gamma} = \gamma_{p,cv}$[5]. □

For $p \in M$, the assignment $v \mapsto \gamma_{p,v}$ defines a map from T_pM to the set of geodesics in M. By Lemma 1.19, we can define a map from a line through the origin in T_pM to a geodesic.

Definition 1.20. Let (M, g) be a Riemannian manifold with an affine connection. The domain of the **exponential map** is defined by

$$E := \{(p, v) \in \mathsf{T}M \mid \gamma_{p,v} \text{ is defined on } [0, 1]\}$$

[4]This follows from existence and uniqueness of solutions to the geodesic equation.
[5]Here $\gamma_{p,cv}$ is the geodesic such that $\gamma_{p,cv}(0) = p$ and $\gamma_{p,cv} = cv$.

and the exponential map on M by

$$\exp : E \to M, \quad v \mapsto \exp(p, v) := \gamma_{p,v}(1)$$

For each $p \in M$, the exponential map of M at p, denoted by $\exp_p$, is the restriction of exp to $E_p := E \cap \mathsf{T}_pM$.

Remark 1.21. *If M is geodesically complete, then $E = \mathsf{T}M$.*

Proposition 1.22. *Let (M, g) be a Riemannian manifold. The exponential map has the following properties:*

(1) *For each $(p, v) \in \mathsf{T}M$, the geodesic $\gamma_{p,v}$ is given by*

$$\gamma_{p,v}(t) = \exp_p(tv)$$

for all t such that either side is defined.

(2) *The set $E_p \subseteq \mathsf{T}_pM$ is star-shaped with respect to 0.*

(3) *E is an open subset of $\mathsf{T}M$ containing the image of the zero section, and the exponential map is smooth.*

(4) *For each point $p \in M$, the differential $d(\exp_p)_0 : T_0(T_pM) \cong T_pM \to T_pM$ is the identity map of T_pM.*

Proof. (**Sketch**) The proof is given below:

(1) This is a simple consequence of Lemma 1.19.

(2) This immediately follows from (1).

(3) See [Lee18, Proposition 5.19].

(4) For any $v \in \mathsf{T}_0(\mathsf{T}_pM) \cong \mathsf{T}_pM$, consider $\tau(t) = tv$. Then

$$d(\exp_p)_0(v) = \left.\frac{d}{dt}\right|_{t=0} (\exp_p \circ \tau)(t) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) = \left.\frac{d}{dt}\right|_{t=0} \gamma_v(t) = v.$$

This completes the proof. □

Remark 1.23. *In general, the differential $(d\exp_p)_v$ is no longer the identity map if v is not the zero tangent vector.*

Example 1.24. The following is a list of examples of exponential maps:

(1) If $v \in \mathsf{T}_p\mathbb{R}^n \cong \mathbb{R}^n$, then the geodesic through p with initial velocity v is given by

$$\gamma_{p,v}(t) = p + tv$$

Hence, $\exp_p(v) = p + v$.

(2) If $v \in \mathsf{T}_p\mathbb{S}^n(R)$, then the geodesic through p with initial velocity v is given by

$$\gamma_{p,v}(t) = \cos atp + \sin(at)\hat{v}, \quad a = |v|/R$$

Hence, $\exp_p(v) = (\cos a)p + (\sin a)\hat{v}$ such that $a = |v|/R$.

We now discuss an important application of the exponential map. Based on Proposition 1.22, the exponential map at every point $p \in M$ is a local diffeomorphism. Hence, there are neighborhoods V of 0 in T_pM and a neighborhood U of p in M such that

$\exp_p : V \to U$ is a diffeomorphism. If V is star-shaped at 0, U is called a normal neighbourhood. Let B be an isomorphism from $\mathbb{R}^n$ to T_pM given by

$$B(x_1, \dots, x_n) = x^i b_i$$

Here (b_i) is an orthonormal basis for T_pM. The (Riemannian) normal coordinates centered at p induced by (b_i) are obtained by combining B with $\exp_p$ to get

$$\phi = B^{-1} \circ (\exp_p|_V)^{-1} : U \to \mathbb{R}^n$$

Proposition 1.25. *Let (M, g) be a Riemannian manifold, and let $(U, (x^i))$ be any normal coordinate chart centered at $p \in M$.*

(1) *The coordinate basis is orthonormal at p.*
(2) *The coordinates of p are $(0, \dots, 0)$.*
(3) *The components of the metric are $g_{ij}(p) = \delta_{ij}$.*
(4) *For every $v = v^i \partial_i|_p \in T_pM$, the geodesic γ_v is represented in normal coordinates by the line*
$$\gamma_v(t) = (tv^1, \dots, tv^n),$$
as long as t is in some interval I containing 0 with $\gamma_v(I) \subseteq U$.
(5) *The Christoffel symbols vanish at p, i.e., $\Gamma^k_{ij}(p) = 0$.*

Exercise 8. Prove Proposition 1.25.

Remark 1.26. *Let $r > 0$ such that $\exp_p$ restricts to a diffeomorphism on $\mathbb{B}_r(0)$. We will call $\exp_p(\mathbb{B}_r(0))$ the geodesic ball of radius r centered at p in M. Similarly, we will call $\partial \exp_p(\mathbb{B}_r(0))$ the geodesic sphere of radius r centered at p in M.*

Remark 1.27. *Geodesics starting at p and lying in a normal neighborhood of p are called radial geodesics.*

2. Geodesics & Metric Balls

Geodesics of Riemannian manifolds were defined in Section 1.2. In a sense, geodesics are acceleration zero curves on a Riemannian manifold. Geodesics of $\mathbb{R}^n$ - which are line segments - enjoy an additional property: these are curves of shortest length between its endpoints. The goal of this section is to propose an alternative characterization of geodesics of Riemannian manifolds as the "shortest" curves in a Riemannian manifold. Along the way, we shall see how geodesic and metric balls are related. Hence, we shall see that geodesics are intimately linked with the underlying metric space topology of a Riemannian manifold.

Remark 2.1. *Let (M, g) be a Riemannian manifold. An admissible curve $\gamma : [a, b] \to I$ is said to be a minimizing curve if*

$$L_g(\gamma) \le L_g(\widetilde{\gamma})$$

for every admissible curve $\widetilde{\gamma}$ with the endpoints $\gamma(a)$ and $\gamma(b)$. One can check that a minimizing curve between two points on a Riemannian manifold. This is best done by using techniques from variational calculus on manifolds. Indeed, if p and q are two points on a

Riemannian manifold, one can define the length function, L_g, on the set of all admissible curves between p and q. Using techniques from the calculus of variations, one can then check that the geodesic equation vcharacterizes the critical points of the length functional. Hence, one can then conclude that a minimizing curve - which must be a a critical point of the length functional - must be geodesic. See [Jos08, Chapter 5] for more details.

The purpose of the remainder of this section is to explore the extent to which the result mentioned in Remark 2.1 is true. It is clear that the full converse of the statement does not hold. We can see this by picking two (not antipodal) points, p, q on a great circle of the unit sphere $\mathbb{S}^n$. There are two geodesics connecting p and q, of which only one is length minimizing. We shall prove in this section that geodesics are locally length minimizing curves.

Definition 2.2. Let (M, g) be a Riemannian manifold. An admissible curve $\gamma : I \to M$ is **locally minimizing** if for every $t_0 \in I$ there exists a neighborhood $I_0 \subseteq I$ containing t_0 such that $\gamma|_{[a,b]}$ is minimizing for every $[a, b] \subseteq I_0$.

The crux of the proof that geodesics are locally minimizing is the Gauss Lemma.

Proposition 2.3. *(**Gauss Lemma**) Let (M, g) be a Riemannian manifold, and let $B_r(0) \subseteq E$ such that $\exp_p$ restricts to a diffeomorphism on $\mathbb{B}_r(0)$. Let $U = \exp_p(B_r(0))$. Let $v, w \in \partial\mathbb{B}_r(0)$. Let $q = \exp_p(v)$.*

(1) *The radial vector field ∂_r is a unit vector field orthogonal to the geodesic spheres in $U \setminus \{p\}$.*
(2) *Let r be the radial distance. Then grad $r = \partial_r$ on $U \setminus \{p\}$.*

Proof. The proof is given below:

(1) See [Lee18, Theorem 6.9] for a proof.
(2) This follows from (2). It suffices to show that ∂_r is orthogonal to the level sets of r and $|\partial_r|_g^2$. The first claim follows directly from (2), and the second from the fact that $\partial_r(r) = 1$ by direct computation in normal coordinates, which in turn is equal to $|\partial_r|_g^2$ by (2).

□

Back to our main goal. We want to prove that geodesics are locally length minimizing. As a first step, we will prove this result for radial geodesics.

Proposition 2.4. *Let (M, g) be a Riemannian manifold. Suppose $p \in M$ and q is contained in a geodesic ball around p. Then (up to reparametrization) the radial geodesic from p to q is the unique minimizing curve in M from p to q.*

Proof. Choose $\epsilon > 0$ such that $U = \exp_p(B_\epsilon(0))$ is a geodesic ball containing q. Let $\gamma : [0, c] \to M$ be the radial geodesic from p to q parametrized by arc length. We have $L_g(\gamma) = c$, since γ has unit speed. Let $\alpha : [0, b] \to M$ be an arbitrary admissible curve from p to q parametrized by arc length. Let $a_0 \in [0, b]$ denote the last time that $\alpha(t) = p$, and $b_0 \in [0,b]$ the first time after a_0 that $\alpha(t)$ meets the geodesic sphere of radius c around p. We have

$$\begin{aligned}
c &= r(\alpha(b_0)) - r(\alpha(a_0)) \\
&= \int_{a_0}^{b_0} \frac{d}{dt} r(\alpha(t))\, dt \\
&= \int_{a_0}^{b_0} dr(\alpha'(t))\, dt \\
&= \int_{a_0}^{b_0} \langle \operatorname{grad} r|_{\alpha(t)}, \alpha'(t)\rangle_g\, dt \\
&\leq \int_{a_0}^{b_0} |\operatorname{grad} r|_{\alpha(t)}|_g |\alpha'(t)|_g\, dt \\
&= \int_{a_0}^{b_0} |\alpha'(t)|_g\, dt \\
&= L_g[\alpha|_{[a_0,b_0]}] \leq L_g[\alpha].
\end{aligned}$$

This shows that γ is a minimizing curve. Now suppose $L_g(\alpha) = c$. Then $b = c$. Both inequalities in the derivation above are equalities. Since α is a unit-speed curve, the second of these equalities implies that $a_0 = 0$ and $b_0 = b = c$, since otherwise the segments of α before $t = a_0$ and after $t = b_0$ would contribute positive lengths. Moreover, the first equality implies that $\alpha'(t)$ is a positive multiple of $\operatorname{grad} r|_{\alpha(t)}$ for each t. Since α is a unit-speed curve, we must have $\alpha'(t) = \operatorname{grad} r|_{\alpha(t)} = \partial_r|_{\alpha(t)}$ for each t. Both α and γ are integral curves of ∂_r passing through q. Hence, $\alpha = \gamma$. □

Corollary 2.5. *Let (M,g) be a connected Riemannian manifold and let $p \in M$.*

(1) *Within every open (or closed) geodesic ball around p, the radial distance function $r(x)$ is equal to the Riemannian distance from p to x in M.*
(2) *Every open or closed geodesic ball is also an open or closed metric ball of the same radius.*
(3) *Every geodesic sphere is a metric sphere of the same radius.*

Proof. The proof is given below:

(1) If x is in the open geodesic ball, the radial geodesic from p to x is minimizing by Proposition 2.4. Since its velocity is equal to ∂_r, which is a unit vector in both the g-norm and the Euclidean norm in normal coordinates, the g-length of γ is equal to its Euclidean length, which is $r(x)$.
(2) See [Lee18, Corollary 6.13] for a proof.
(3) See [Lee18, Corollary 6.13] for a proof.

□

Remark 2.6. *Note that we have yet to fully prove the statement that geodesics are locally length minimizing curves. Indeed, Proposition 2.4 only deals with the case of radial* *geodesics. However, discussion above is sufficient for our discussion of the Hopf-Rinow theorem in the next section. Refer to [Lee18, Proposition 6.14] for a proof of the complete statement that geodesics are locally length minimizing curves.*

3. Hopf-Rinow Theorem

In the previous section, we have uncovered a relationship between the metric properties of a Riemannian manifold and the geometric properties of a Riemannian manifold. Indeed, we have seen that geodesics are locally length minimizing curves. Moreover, we have seen that open (closed) geodesic balls are open (closed) metric balls. We now further this relationship of such properties by proving the Hopf-Rinow theorem, which asserts that a Riemannian manifold is geodesically complete if and only if the underlying metric space is (metrically) complete. We shall need the following lemma.

Lemma 3.1. *Let (M, g) be a connected Riemannian manifold. Suppose there exists a point $p \in M$ such that the restricted exponential map $\exp_p$ is defined on all of T_pM. For any $q \in M$, there is a unit-speed minimizing geodesic from p to q.*

Remark 3.2. *Clearly, the converse of Lemma 3.1 is not true. Consider $\mathbb{B}^n$. Any two points can be joined by a geodesic, but the restricted exponential map $\exp_p$ need not be define on on all of T_pM for $p \in M$ "sufficiently close" to $\mathbb{S}^{n-1}$.*

Remark 3.3. *We sketch the main idea behind the proof of Lemma 3.1.*

- *Start with a normal neighborhood $U = \exp_p(\mathbb{B}_\delta(0))$ of p. By Proposition 1.25, the geodesics from p are of the form $\exp_p(sv)$ for some v in T_pM with $|v| = 1$ and $s < \delta$.*
- *Fix a point x_0 on the geodesic sphere $\exp_p(\partial\mathbb{B}_\delta(0))$ that minimizes $d(x, q)$ amongst all x on the on the geodesic sphere $\exp_p(\partial\mathbb{B}_\delta(0))$. Then our desired γ us a radial geodesic from p to x_0.*
- *Repeat this process from x_0 until we get to q. At each step, we prove that the geodesic segment added, in fact, coincides with γ.*

Proof. (**Sketch**) Let $\varepsilon > 0$ be small enough such that the closed geodesic ball $U = \exp_p(\overline{\mathbb{B}}_\varepsilon(0))$ doesn't contain q. Since ∂U is compact, there is a point x on ∂U that minimizes the function $d_g(q, \cdot)$ on ∂U. Let γ be a unit-speed minimizing radial geodesic such that $\gamma_{[0,\varepsilon]}$ connects p and x[6].

We claim that γ is such that

$$d_g(p, q) = d_g(p, \gamma(\varepsilon) + d_g(\gamma(\varepsilon), q) = d_g(p, x) + d_g(x, q) \tag{3}$$

Let $\sigma : [c, d] \in M$ be any admissible curve from p to q. Let t_0 be the first time σ hits ∂U, and let σ_1 and σ_2 denote the restrictions of σ to $[c, t_0]$ and $[t_0, d]$, respectively. Since every point in ∂ is at a distance ε from p, we have

$$L_g(\sigma_1) \geq d_g(p, \sigma(t_0)) = d_g(p, x)$$

Similarly, we have

$$L_g(\sigma_1) \geq d_g(\sigma(t_0), q) \geq d_g(x, q)$$

[6]Note that γ is defined for all $t \in \mathbb{R}$.

Altogether, we have,

$$L_g(\sigma) = L_g(\sigma_1) + L_g(\sigma_2) \geq d_g(p, x) + d_g(x, q)$$

Taking infimum over all such curves, we have

$$d_g(p, g) \geq d_g(p, x) + d_g(x, q)$$

Moreover, we have by the triangle inequality,

$$d_g(p, g) \leq d_g(p, x) + d_g(x, q)$$

Therefore, we have

$$d_g(p, g) = d_g(p, x) + d_g(x, q)$$

We say that $\gamma|_{[0,\varepsilon]}$ aims at q since it satisfies Equation (3)[7]. If $T = d_g(p, q)$, we wish to show that

$$z := \sup\{b \in [0, T] : \gamma_{[0,b]} \text{ aims at } q\} = T$$

We have already seen that $z \geq \varepsilon$. It can be shown that indeed $z = T$[8]. This is sufficient to complete the proof. Since γ is minimizing, we have that

$$d_g(p, \gamma(T)) = L_g(\gamma) = d(p, q)$$

Therefore, we have,

$$d_g(p, q) = d_g(p, \gamma(T)) + d_g(\gamma(T), q) = d_g(p, q) + d_g(\gamma(T), q)$$

Hence, $d_g(\gamma(T), q) = 0$ which implies that $\gamma(T) = q$. This shows that $\gamma|_{[0,T]}$ is a unit-speed minimizing geodesic from p to q. □

Exercise 9. Complete the proof of Lemma 3.1. Or see [Lee18, Lemma 6.18][9].

Theorem 3.4. *(**Hopf-Rinow Theorem**) Let (M, g) be a connected Riemannian manifold. Then the following conditions are equivalent:*

(1) *M is complete as a metric space.*
(2) *M is geodesically complete.*

Proof. The proof is given below:

- (1) implies (2): Assume M is a complete metric but M is not geodesically complete. There is some unit-speed maximal geodesic $\gamma : I \to M$ such that $I \neq \mathbb{R}$. Since γ is a maximal geodesic, I is an open interval of the form (a, b) containing 0. If $b < \infty$, let $\{t_i\}$ be any increasing sequence in (a, b) that converges to b. Set $q_i = \gamma(t_i)$. Since γ is parametrized by arc length, the length of $\gamma|_{[t_i,t_j]}$ is exactly $|t_j - t_i|$, so

$$d_g(q_i, q_j) = |t_j - t_i|$$

[7] In a sense, this means that $\gamma|_{[0,\varepsilon]}$ is parallel to the straight line that would connect p and q if we were working in Euclidean space.

[8] In a sense, this means that $\gamma|_{[0,T]}$ coincides with the straight line that would connect p and q if we were working in Euclidean space.

[9] Some details of the proof have been left out due to time constraints.

Hence, (q_i) is a Cauchy sequence in M. By completeness, (q_i) converges to some point $q \in M$. Let W be a uniformly δ-normal neighborhood of q for some $\delta > 0$[10]. Choose j large enough that $t_j > b - \delta$ and $q_j \in W$. The fact that $B_\delta(q_j)$ is a geodesic ball means that every unit-speed geodesic starting at q_j exists at least for $t \in (a, \delta]$. In particular, this is true of the geodesic α with $\alpha(0) = q_j$ and $\dot\alpha(0) = \dot\gamma(t_j)$ for $t \in [0, \delta]$. Define $\widetilde{\gamma} : [0, t_j + \delta] \to M$ by

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{for } t \in (a, b), \\ \alpha(t - t_j) & \text{for } t \in (t_j - \delta, t_j + \delta). \end{cases}$$

The expression on the right hand side are geodesics and their position and velocity agree at t_j. Hence, the two geodesics agree on the overlap of the domains of the geodesics. Since $t_j + \delta > \delta > b$, $\widetilde{\gamma}$ is an extension past b, which is a contradiction. Hence, $b = +\infty$. A similar argument shows that $a = -\infty$. Hence, M must be geodesically complete.

- (2) implies (1): Let (q_i) be a Cauchy sequence in M. Since M is geodesically complete, we have that $\exp_p$ is defined on all of T_pM for each $p \in M$. Therefore, Lemma 3.1 implies that any two points of M can be joined by a unit-speed minimizing geodesic. For each i, let $\gamma_i(t) = \exp_p(tv_i)$ be a unit-speed minimizing geodesic from p to q_i. Let $d_i = d_g(p, q_i)$. We have

$$\exp_p(d_i v_i) = \gamma_{d_i v_i}(1) = \gamma_{v_i}(d_i) = \gamma_{v_i}(d_g(p, q_i)) = q_i$$

Clearly, d_i is a bounded sequence. Since $|v_i| = 1$, the sequence $(d_i v_i)$ of vectors in T_pM is bounded. Thus, a subsequence $(d_{i_k} v_{i_k})$ converges to some $v \in \mathsf{T}_pM$, and by continuity of the exponential map

$$\lim_{k\to\infty} \exp_p(d_{i_k} v_{i i_k k}) = \exp_p(v)$$

Since the original sequence $(q_i = exp_p(d_i v_i))$ is a Cauchy sequence, we must have that $(q_i = exp_p(d_i v_i))$ converges to the limit $q = \exp_p(v)$ in M.

□

A connected Riemannian manifold is simply said to be complete if it is either geodesically complete or metrically complete. Theorem 3.4 then implies that it is both these properties are equivalent. We end our discussion by discussing some corollaries of the Hopf-Rinow theorem.

Corollary 3.5. *Let (M, g) be a connected Riemannian manifold.*

(1) *If there exists a point $p \in M$ such that the restricted exponential map $\exp_p$ is defined on all of T_pM, then M is complete.*
(2) *If M is complete, then any two points in M can be joined by a unit-speed minimizing geodesic segment.*
(3) *If M is compact, then M is complete.*

[10]This means that W is contained in a geodesic ball of radius δ around each of its points. See [Lee18, Lemma 5.14] for a proof that such neighbourhoods exist.

(4) *If M is a homogeneous Riemannian manifold, then M is complete.*
(5) *If M is a Riemannian symmetric space, then M is complete.*

Proof. The proof is given below:

(1) follows directly from the the observation that the hypothesis was used to proved that M is metrically complete in the proof of (2) implies (1) in Theorem 3.4.
(2) A complete Riemannian manifold is geodesically complete. Hence, the restricted exponential map $\exp_p$ is defined on all of T_pM for each $p \in M$. The desired statement now follows from Lemma 3.1.
(3) A compact metric space is a complete metric space. The claim now follows from Theorem 3.4.
(4) Assume that M is not geodesically complete. There is some unit-speed maximal geodesic $\gamma : I \to M$ such that $0 \in I = (a, b) \neq \mathbb{R}$. Let $p = \gamma(0)$. If $b < \infty$, let $\{t_i\}$ be any increasing sequence in (a, b) that converges to b. Set $q_i = \gamma(t_i)$. Let $\delta > 0$ such that $U = \exp_p(B_\delta(0))$ is a geodesic ball centered at p. Choose j large enough such that $t_j > b - \delta$. Since M is a homogeneous Riemannian manifold, there exist $F_j \in \mathbf{Isom}(M, g)$ such that $F_j(p) = q_j$. Consider the following diagram:

$$\begin{tikzcd} M \arrow[r, "F_j"] & M \\ \mathsf{T}_pM \arrow[u, "\exp_p"] \arrow[r, "dF_j"] & \mathsf{T}_qM \arrow[u, "\exp_q"'] \end{tikzcd}$$

The diagram commutes due to the naturality of the exponential map. Refer to [Lee18, Proposition 5.20]. Since F_j is an isometry, dF_j is a linear isometry. Hence, $dF_j(B_\delta(0))$ is isometric to $B_\delta(0)$. Therefore, $\exp_q(dF_j(B_\delta(0)))$ can be identified with a geodesic ball of radius δ centered at q. Since the diagram commutes, we have that $F_j(U)$ is a geodesic ball of radius δ centered at q_j. Now as in the proof of the statement (1) implies (2) in Theorem 3.4, we can construct $\widetilde{\gamma}$ that extends γ past b. Hence, we must have $b = +\infty$. Similarly, we must have $a = -\infty$. This shows that M is geodesically complete, and hence complete.
(5) If a geodesic γ is defined on $[0, s)$, we may reflect it by $s|_{\gamma(t)}$ for some $t \in (s/2, s)$, hence we may extend it beyond s. Hence, M is geodesically complete.

□

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