# MATH 740 HOMEWORK

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Abstract. These are solutions to the homework problems I graded for the course *MATH 740: Fundamental Concepts of Differential Geometry* at Maryland. Solution to assignment three is missing.

Contents



## Notation

In this section, I briefly summarize the notation used in this document:

(1) The components of $v \in \mathbb{R}^n$ are written as $v^1, \cdots, v^n$ with upper indices.
(2) The basis vectors of $\mathbb{R}^n$ are written as $e_1, \cdots, e_n$ with lower indices.
(3) The partial derivative operation on $\mathbb{R}^n$ is written as either $\partial_i$ or $\partial_i^{\mathbb{R}^n}$. On occasion, we will write the partial derivative operators as

$$\frac{\partial}{\partial x^i}\bigg|_p$$

for $p \in \mathbb{R}^n$ to make arguments more explicit when necessary.
(4) The Einstein summation convention is used throughout. For example,

$$v^i e_i \text{ is an abbreviation for } \sum_{i=1}^{n} v^i e_i$$

(5) For a smooth manifold, $M$, and $p \in M$, the basis vectors for $\mathsf{T_p}M$ are written as $\partial_i^M$. When no confusion arises, the basis vectors will be written as simply $\partial_i$. In local coordinates, we shall continue to write the basis vectors as $\partial_i^M$ or $\partial_i$. When deemed necessary, we will write the basis vectors as

$$\frac{\partial}{\partial x^i}\bigg|_p$$

for $p \in M$ in local coordinates.

(6) For a smooth manifold, $M$, and $p \in M$, the basis vectors for $\mathsf{T}^*_\mathsf{p}M$ are written as $\varepsilon^i_M$. When no confusion arises, the basis vectors will be written as simply $\varepsilon^i$. In coordinates, the basis vectors will usually be written as $dx^i|_p$.

# MATH 740 HOMEWORK 1

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A good general reference is *Introduction to Smooth Manifolds,* by John M. Lee, ISBN 978-1-4899-9475-2, Springer Graduate Texts in Mathematics 218, Second Edition (2012).

**Problem 1.** Let $R > 0$ and let $X$ be a sphere of radius $R > 0$. Suppose that $p \in X$ and let $r > 0$. The *metric circle* $C_r(p)$ consists of all points $q \in X$ with distance $d(p, q) = r$ (respectively $d(p, q) < r$). Show that

$$\begin{aligned} \mathsf{length}\ C_r(p) &= 2\pi R \sin(r/R) \\ &= 2\pi r - \frac{\pi}{3} R^{-2} r^3 + \dots \end{aligned}$$

It bounds the *metric disc* $D_r(p)$ comprising points $q \in X$ with distance $d(p, q) < r$) and

$$\begin{aligned} \mathsf{area}\ D_r(p) &= 2\pi R^2 \big(1 - \cos(r/R)\big) \\ &= \pi r^2 - \frac{\pi}{12} R^{-2} r^4 + \dots \end{aligned}$$

Thus intrinsic measurements on $X$ detect the variable $R$. The quantity $R^{-2}$ is the *Gaussian curvature* of $X$.

**Problem 2.** Let $X$ be the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. Define the *Poincaré metric,* by the metric tensor

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

- The group of all transformations $(x, y) \mapsto (Ax + b, Ay)$, where $A > 0, b \in \mathbb{R}$, acts isometrically on $(X, ds^2)$.
- It is convenient to use complex notation, replacing $(x, y) \in \mathbb{R}^2$ by $z := x + iy \in \mathbb{C}$. Then *reflection* in the unit semicircle $\iota$

  $$\begin{aligned} z &\stackrel{\iota}{\longmapsto} 1/\overline{z} \\ (x, y) &\mapsto \big(x/(x^2 + y^2), y/(x^2 + y^2)\big) \end{aligned}$$

  also acts isometrically on $(X, ds^2)$.

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- Show that these transformations generate a group,[1] isomorphic to $\mathsf{PGL}(2,\mathbb{R})$, consisting of all isometries of the Riemannian manifold $(X, ds^2)$. The subgroup of *orientation-preserving* isometries identifies with the group $\mathsf{PSL}(2,\mathbb{R})$ consisting of real linear fractional transformations
$$z \longmapsto \frac{az+b}{cz+d}$$
(where $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$), acting on the upper half plane $\{z \in \mathbb{C} \mid \mathsf{Im}(z) > 0\}$.
- In this group, the *involution* or *symmetry* in the point in $X$ corresponding to the imaginary unit $i$ (and the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$) is the linear fractional transformation $z \xmapsto{\sigma} -1/z$. Prove that, similarly to the calculation with $\iota$ above, $\sigma^* ds^2 = ds^2$.

**Problem 3.** Let $A$ be an $\mathbb{R}$-algebra and $A \xrightarrow{\varepsilon} \mathbb{R}$ an $\mathbb{R}$-algebra homomorphism. A *derivation* of $A$ over $\varepsilon$ is an $\mathbb{R}$-linear map $A \xrightarrow{D} \mathbb{R}$ such that
$$D(fg) = D(f)\varepsilon(g) + \varepsilon(f)D(g)$$
for all $f, g \in A$.

Let $X$ be an open neighborhood of $p \in \mathbb{R}^n$.

- Let $A = \mathsf{C}^\infty(X)$ be the algebra of smooth functions on $X$ and $\varepsilon$ be evaluation at $p$:
$$\begin{aligned} A &\xrightarrow{\varepsilon} \mathbb{R} \\ f &\longmapsto f(p) \end{aligned} \tag{1}$$
Show that the the derivations of $A$ over $\varepsilon$ form a vector space with basis the partial derivative operators at $p$:
$$\left.\frac{\partial}{\partial x^i}\right|_p : f \longmapsto \frac{\partial f}{\partial x^i}(p)$$
for $i = 1, \dots, n$.
- Let $A = \mathsf{C}(X)$ (continuous functions on $X$ and $\varepsilon$ be evaluation at $p$ as in (1). Show that no nonzero derivations of $A$ over $\varepsilon$ exist.

[1]The symbol $\mathsf{P}$ denotes *projectivization,* that is, *projective equivalence classes* of matrices: where two matrices are equivalent if they differ by multiplication by a nonzero scalar.

**Problem 4.** Let $X$ be a smooth $n$-manifold with cotangent bundle $\mathsf{T}^*X \xrightarrow{\Pi_X} X$. Let $\mathsf{Diff}(X)$ denote the group of diffeomorphisms $X \to X$.

- Define the smooth action of $\mathsf{Diff}(X)$ on $\mathsf{T}^*X$ and prove that this action preserve the fibration $\Pi_X$ and in particular the *zero-section* $\mathbf{0}_X$ of $\mathsf{T}^*X$.
- Construct a covector field (that is, a 1-form) $\alpha$ on $\mathsf{T}^*X$ which is invariant under $\mathsf{Diff}(X)$.
- Construct an action of the semidirect product $G := \mathsf{Diff}(X) \ltimes \mathsf{C}^\infty(X)$ on $\mathsf{T}^*X$ which extends the action of $\mathsf{Diff}(X)$ and preserves the covector field $\alpha$, where $\mathsf{C}^\infty(X)$ is the vector group with the natural action of $\mathsf{Diff}(X)$.
- Show that the exterior derivative $d\alpha$ is a closed everywhere nondegenerate exterior 2-form (that is, a *symplectic structure*) which is invariant under the action of $G$.

**Problem 5.** Recall the Lie groups $\mathsf{GL}(n, \mathbb{R}), \mathsf{SL}(n, \mathbb{R}), \mathsf{O}(n)$: The *general linear group* consists of all invertible $n \times n$ matrices. The *special linear group* consists of all $n \times n$ matrices of determinant one (sometimes called *unimodular*). The *orthogonal group* consists of all orthogonal $n \times n$ matrices.

- $\mathsf{GL}(n, \mathbb{R}), \mathsf{SL}(n, \mathbb{R}), \mathsf{O}(n)$ are all smooth submanifolds of the vector space of $n \times n$-matrices. Compute their dimensions.
- Identify which of these groups is compact.
- $\mathsf{GL}(n, \mathbb{R})$ has two components and $\mathsf{SL}(n, \mathbb{R})$ is connected.
- The *special orthogonal group* $\mathsf{SO}(n) := \mathsf{SL}(n, \mathbb{R}) \cap \mathsf{O}(n)$ is the identity component of $\mathsf{O}(n)$.
- Every element of $\mathsf{SO}(n)$ is a rotation if and only if $n < 4$.
- $\mathsf{SO}(2) \approx S^1$ and $\mathsf{SO}(3) \approx \mathbb{R}\mathsf{P}^3$.

## 1. Homework 1

**Exercise 1.** Denote a sphere of radius $R$ as $\mathbb{S}^2_R$. Fix $p \in \mathbb{S}^2_R$ and let $0 < r < R\pi$.

(1) We can assume that the point, $p$, is the north pole, $N = (0, 0, R)$ because $\mathbf{SO}(3, \mathbb{R})$, acts transitively and isometrically on $\mathbb{S}^2_R$.
(2) We consider the standard parameterization of $\mathbb{S}^2_R$ in spherical coordinates.
(3) The *metric circle*, $C_r(N)$, and the *metric disc*, $D_r(N)$, are described as the following subsets:

$$C_r(N) = \Gamma([0, 2\pi) \times \{r/R\}) \qquad D_r(N) = \Gamma([0, 2\pi) \times [0, r/R])$$

- $C_r(N)$ is the great circle bounding the spherical cap, $D_r(N)$, on $\mathbb{S}^2_R$. The great circle is the intersection of the plane

$$\{(x, y, z) \in \mathbb{R}^3 \ : \ z = R\cos(r/R)\}$$

with $\mathbb{S}^2_R$. The great circle is then described by the equation,

$$x^2 + y^2 + (R\cos(r/R))^2 = R^2,$$

implying that

$$x^2 + y^2 = R^2 - R^2\cos^2(r/R) = R^2\sin^2(r/R) = (R\sin(r/R))^2$$

Hence, $C_R(N)$ is a great circle of radius $R\sin(r/R)$ Therefore:

$$\textbf{length}\ C_r(N) = 2\pi R\sin(r/R) = 2\pi r - \frac{\pi r^3}{3R^2} + O(r^5)$$

- $D_r(N)$ is a spherical cap $\mathbb{S}^2_R$. The spherical cap can be generated by by revolving the graph of

$$f(x) = \sqrt{R^2 - x^2} \qquad x \in [0, R\sin(r/R)]$$

about the $y$-axis. We then have the following:

$$\begin{aligned}
\textbf{area } D_r(N) &= 2\pi \int_0^{R\sin(r/R)} x\sqrt{1+f'(x)^2}dx, \\
&= 2\pi \int_0^{R\sin(r/R)} x\sqrt{1+\frac{x^2}{R^2-x^2}}dx, \\
&= 2\pi \int_0^{R\sin(r/R)} x\sqrt{\frac{R^2}{R^2-x^2}}dx, \\
&= 2\pi R(R-\sqrt{R^2-R^2\sin^2(r/R)}, \\
&= 2\pi R^2(1-\sqrt{1-\sin^2(r/R)}, \\
&= 2\pi R^2(1-\cos(r/R)), \\
&= 2\pi R^2\left(\frac{r^2}{2R^2}-\frac{r^4}{24R^4}+O(r^6)\right) = \pi r^2 - \frac{\pi r^4}{12R^2}+O(r^6)
\end{aligned}$$

**Exercise 2.** Let $X$ be the upper half-plane

$$X = \{(x,y)\in\mathbb{R}^2 \mid y>0\}$$

endowed with the Poincaré metric,

$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

We state a relevant result below about pullbacks of tensor fields below adapted in the context of our problem:

Let $F : X \to X$ be a smooth map. Assume that the domain is described by co-ordinates $(x,y)$ and the co-domain is described by co-ordinates $(x,y)$. Then $F$ can be written as:

$$F : X \to X \qquad (F^1(x,y), F^2(x,y)) := (u(x,y), v(x,y))$$

Then $F^*ds^2$ has the following expression:

$$\begin{aligned}
F^*ds^2 &= F^*\left(\frac{du^2+dv^2}{v^2}\right) \\
&= \frac{1}{(F^2(x,y))^2}\cdot(dF^1(x,y))^2 + \frac{1}{(F^2(x,y))^2}\cdot(dF^2(x,y))^2
\end{aligned}$$

In other words, compute the differential of $F^1(x,y)$ and $F^2(x,y)$ and multiply each differential with $1/[F^2(x,y)]^2$.

- Consider

$$T_{a,b} : X \to X \qquad T_{a,b}(x,y) = (ax+b, ay)$$

where $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. $T_{a,b}$ is clearly invertible since its inverse is given by:

$$T_{a^{-1},-a^{-1}b} : X \to X \qquad T_{a^{-1},-a^{-1}b}(u,v) = (a^{-1}(u-b), a^{-1}v)$$

Clearly, both $T_{a,b}$ and $T_{a^{-1},-a^{-1}b}$ are smooth maps. Therefore, $T_{a,b}$ is a diffeomorphism. We show that $T^*_{a,b}ds^2 = ds^2$.

$$\begin{aligned} T^*_{a,b}ds^2 &= \frac{(adx)^2 + (ady)^2}{a^2y^2}, \\ &= \frac{a^2dx^2 + a^2dy^2}{a^2y^2}, \\ &= \frac{dx^2 + dy^2}{y^2} = ds^2. \end{aligned}$$

Hence, the group $G = \{T_{a,b} : a \in \mathbb{R}^+, a > 0\}$[1] acts isometrically on $X$.

- Consider

$$\iota : X \to X \qquad \iota(z) = \frac{1}{\bar{z}}$$

$\iota$ is clearly invertible since it is an idempotent map. Clearly, $\iota$ is smooth as well because it can be written as

$$\iota : X \to X \qquad \iota(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

We show that $\iota^*ds^2 = ds^2$.

$$\iota^*ds^2 = \frac{(x^2+y^2)^2}{y^2}\left(\left[\frac{y^2-x^2}{(x^2+y^2)^2}dx - \frac{2xy}{(x^2+y^2)^2}dy\right]^2 + \left[\frac{x^2-y^2}{(x^2+y^2)^2}dy - \frac{2xy}{(x^2+y^2)^2}dx\right]^2\right)$$

The first term in parenthesis expands to:

$$\frac{(y^2-x^2)^2}{(x^2+y^2)^4}dx^2 + \frac{4x^2y^2}{(x^2+y^2)^4}dy^2 - \frac{2xy(y^2-x^2)}{(x^2+y^2)^4}dxdy$$

The second term in parenthesis expands to:

$$\frac{4x^2y^2}{(x^2+y^2)^4}dx^2 + \frac{(x^2-y^2)^2}{(x^2+y^2)^4}dy^2 - \frac{2xy(x^2-y^2)}{(x^2+y^2)^4}dxdy$$

[1]It is clear that this is a group. For instance, $T_{1,0}$ is the identity. We also have $T_{a,b} \circ T_{c,d} = T_{ac,bc+d}$. Moreover, $T^{-1}_{a,b} = T_{a^{-1},-a^{-1}b}$.

It is clear that the cross terms cancel. We have:

$$\begin{aligned}
\iota^* ds^2 &= \frac{(x^2+y^2)^2}{y^2}\left(\frac{(x^2-y^2)^2+4x^2y^2}{(x^2+y^2)^4}dx^2 + \frac{(x^2-y^2)^2+4x^2y^2}{(x^2+y^2)^4}dy^2\right)\\
&= \frac{(x^2+y^2)^2}{y^2}\left(\frac{(x^2+y^2)^2}{(x^2+y^2)^4}dx^2 + \frac{(x^2+y^2)^2}{(x^2+y^2)^4}dy^2\right)\\
&= \frac{(x^2+y^2)^2}{y^2}\left(\frac{1}{(x^2+y^2)^2}dx^2 + \frac{1}{(x^2+y^2)^2}dy^2\right)\\
&= \frac{dx^2+dy^2}{y^2} = ds^2
\end{aligned}$$

Hence, $\iota$ acts isometrically on $X$.

- We shall make use of the following fact from complex analysis[2]:

Let $\mathbb{B}$ denote the unit disk in $\mathbb{R}^2$ which is identified with $\mathbb{C}$. Then $F : X \to \mathbb{B}$ $\quad F(z) = \frac{z-i}{z+i}$ is a holomorphic map with holomorphic inverse given by $G : \mathbb{B} \to X$ $\quad G(w) = i \cdot \frac{1-w}{1+w}$

A simple calculation shows that $F'(z) \neq 0$ for each $z \in X$. Therefore, $F$ is a holomorphic, conformal transformation. $F$ is also orientation-preserving since $F$ is a (locally) invertible holomorphic map. Clearly, $F$ is also a diffeomorphism. Therefore, one can define a metric on $\mathbb{B}^2$ by $F^*(ds^2)$ such that $(X, ds^2)$ and $(\mathbb{B}^2, F^*(ds^2))$ are isometric (by definition). $(\mathbb{B}^2, F^*(ds^2))$ is called the *Poincaré disk model* for the hyperbolic plane. The task of computing the isometries of $X$ is equivalent to the task of computing the isometries of the Poincaré disk. To achieve the latter goal, we shall make use of an additional fact from complex analysis:

A map $f$ is an automorphism of $\mathbb{B}^2$ if and only $f_{a,b}(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ for $a, b \in \mathbb{C}$ such that $|a|^2 - |b|^2 = 1$

This result is a consequence of Schwarz's lemma. One can check that each $f_{a,b}$ is orientation-preserving. Armed with the previous two observations, we have that $\mathbf{Aut}(\mathbb{B}) \cong \mathbf{Aut}^+(X)$ with the explicit isomorphism given by

$$\Phi : \mathbf{Aut}(\mathbb{B}) \to \mathbf{Aut}^+(X) \qquad \Phi(f_{a,\theta}) = G \circ f_{a,b} \circ F$$

Here $\mathbf{Aut}^+(X)$ is the group of orientation preserving isometries of $X$. A somewhat lengthy calculation which we omit shows that an element of

[2]Any proof classifying isometries of the hyperbolic plane shall make use of some non-trivial result.

$\mathsf{Aut}^+(X)$ is of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $a, b, c,$ and $d$ are real, and $ad-bc = 1$. Since $F$ and $G$ are orientation-preserving, we have that

$$\mathsf{Aut}_2^+(X) = \left\{ z \mapsto \frac{az+b}{cz+d} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2, \mathbb{R}) \right\}$$

In fact, we have that

$$\mathsf{Aut}^+(X) = \mathrm{Isom}^+(X)$$

It suffices to show that $\mathsf{Aut}^+(X) \subseteq \mathrm{Isom}^+(X)$. That is, every transformation of the form

$$z \mapsto \frac{az+b}{cz+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2, \mathbb{R})$$

is an isometry of $X$. This is a simple but tedious calculation which we skip. Hence, we obtain a surjective homomorphism

$$\Gamma : \mathsf{SL}(2, \mathbb{R}) \to \mathrm{Isom}^+(X) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$$

One can check that

$$\ker \Gamma = \{\lambda I_2 : \lambda \in \mathbb{R}^\times\}$$

Hence,

$$\mathrm{Isom}^+(X) \cong \mathsf{PSL}(2, \mathbb{R})$$

Consider the short exact sequence

$$1 \longrightarrow \mathsf{SL}(2, \mathbb{R}) \hookrightarrow \mathsf{GL}(2, \mathbb{R}) \xrightarrow{\det} \mathbb{R}^* \longrightarrow 1$$

This short exact sequence descends to a short exact sequence

$$1 \longrightarrow \mathsf{PSL}(2, \mathbb{R}) \hookrightarrow \mathsf{PGL}(2, \mathbb{R}) \xrightarrow{\det} \mathbb{R}^*/\mathbb{R}^+ \longrightarrow 1$$

In particular, we have that

$$|\mathsf{PGL}(2, \mathbb{R}) : \mathsf{PSL}(2, \mathbb{R})| = |\mathbb{R}^*/\mathbb{R}^+| = 2.$$

By the next part, we know that there isn orientation reversing symmetry, $\sigma$, of $X$ contained in $\mathsf{PGL}(2, \mathbb{R}) \setminus \mathsf{PSL}(2, \mathbb{R})$. It is then clear that all elements of the set

$$\sigma \mathsf{PSL}(2, \mathbb{R})$$

consists of orientation-reversing isometries. This shows that

$$\mathrm{Isom}(X) \cong \mathsf{PGL}(2, \mathbb{R})$$

- Consider

$$\sigma : X \to X \qquad \sigma(z) = -\frac{1}{z}$$

$\sigma$ is clearly invertible since it is an idempotent map. Clearly, $\sigma$ is smooth as well because it can be written as

$$\sigma : X \to X \qquad \sigma(x,y) = \left(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

We show that $\sigma^* ds^2 = ds^2$.

$$\sigma^* ds^2 = \frac{(x^2+y^2)^2}{y^2}\left(\left[\frac{x^2-y^2}{(x^2+y^2)^2}dx + \frac{2xy}{(x^2+y^2)^2}dy\right]^2 + \left[\frac{x^2-y^2}{(x^2+y^2)^2}dy - \frac{2xy}{(x^2+y^2)^2}dx\right]^2\right)$$

The first term in parenthesis expands to:

$$\frac{(x^2-y^2)^2}{(x^2+y^2)^4}dx^2 + \frac{4x^2y^2}{(x^2+y^2)^4}dy^2 + \frac{4xy(x^2-y^2)}{(x^2+y^2)^2}dxdy$$

The second term in parenthesis expands to:

$$\frac{4x^2y^2}{(x^2+y^2)^4}dx^2 + \frac{(x^2-y^2)^2}{(x^2+y^2)^4}dy^2 - \frac{4xy(x^2-y^2)}{(x^2+y^2)^4}dxdy$$

It is clear that the cross terms cancel. We have:

$$\begin{aligned}\sigma^* ds^2 &= \frac{(x^2+y^2)^2}{y^2}\left(\frac{(x^2-y^2)^2+4x^2y^2}{(x^2+y^2)^4}dx^2 + \frac{(x^2-y^2)^2+4x^2y^2}{(x^2+y^2)^4}dy^2\right) \\ &= ds^2\end{aligned}$$

as above. Hence, $\sigma$ acts isometrically on $X$.

This completes the proof.

**Exercise 3.** Fix $p \in \mathbb{R}^n$ and let $\varepsilon$ be the evaluation map at $p$. Let $A = \mathsf{C}^\infty(\mathbb{R}^n)$ or $A = \mathsf{C}(\mathbb{R}^n)$ and let $\mathrm{Der}_p(\mathbb{R}^n)$ denote the space of derivations of $A$ over $\varepsilon$. For some $X_p \in \mathrm{Der}_p(\mathbb{R}^n)$, let us first record two key properties of $X_p$ that follow immediately from the definition:

(1) For any constant function $f \in A$, $X_p(f) = 0$. Indeed, note that:

$$X_p(1) = X_p(1 \cdot 1) = 1 \cdot X_p(1) + X_p(1) \cdot 1 = 2X_p(1)$$

Hence, $X_p(1) = 0$. If $f = c$ for some constant $c \in \mathbb{R}$, then $f = c \cdot 1$ where 1 is the constant function taking value 1. Thus, by linearity

$$X_p(f) = X_p(c \cdot 1) = cX_p(1) = 0$$

(2) If $f, g \in A$ with $f(p) = g(p) = 0$, then $X_p(fg) = 0$. Indeed,

$$X_p(fg) = f(p)X_p(g) + X_p(f)g(p) = 0 + 0 = 0$$

- Let $A = \mathrm{C}^\infty(\mathbb{R}^n)$. Let $\mathsf{T}_p\mathbb{R}^n$ be the geometric tangent space of $\mathbb{R}^n$ at $p$:

$$\mathsf{T}_p\mathbb{R}^n = \{p\} \times \mathbb{R}^n := \{v_p \in \mathbb{R}^n | v \in \mathbb{R}^n\}$$

Moreover, let $D_{v_p}$ denote the directional derivative operator at $p$ in the direction of $v$. We show that the map

$$\Phi_p : \mathsf{T}_p\mathbb{R}^n \to \mathrm{Der}_a\mathbb{R}^n \qquad v_p \mapsto D_{v_p}$$

is a vector space isomorphism. If $v_p, w_p \in \mathsf{T}_p\mathbb{R}^n, c \in \mathbb{R}$ and $f \in A$, we have that:

$$D_{cv_p+w_p}(f) = cv^i\partial_i f + w^i\partial_i f = cD_{v_p}(f) + D_{w_p}(f)$$

Hence, $\Phi_p$ is a linear map.

Let $(a, v) \in \ker \Phi_p$. Hence, so that $D_{v_p} f = 0$ for all $f \in \mathrm{C}^\infty(\mathbb{R}^n)$. Taking $f(x) = x^j$ for $1 \leq j \leq n$ to be the $j$-th coordinate function, we have

$$0 = D_{v_p}(x^j) = v^i\partial_i x^j(p) = v^i\delta_i^j = v^j.$$

This shows that $v = 0$, and so $\Phi_p$ is one-to-one.

Fix $X_p \in \mathrm{Der}_p\mathbb{R}^n$. Define the vector $v$ by taking its $j$-th component in the standard basis to be $X_p(x^j)$. That is, let $v^j = X_p(x^j)$. Let $f \in \mathrm{C}^\infty(\mathbb{R}^n)$. By Taylor's Theorem,

$$f(x) = f(p) + \partial_i f(p)(x^i - p^i) + \frac{1}{2}(x^i - p^i)(x^j - p^j)\int_0^1 (1-t)\partial_i\partial_j f(a + t(x-p))dt$$

Setting

$$g^i(x) = (x^i - p^i), \qquad h^j(x) = (x^j - p^j)\int_0^1 (1-t)\partial_i\partial_j f(a + t(x-p)),$$

the functions $g^i, h^j$ are smooth and satisfy $g^i(p) = h^j(p) = 0$ for $1 \leq i, j \leq n$, and we have

$$f(x) = f(p) + \partial_i f(p)(x^i - p^i) + \frac{1}{2}\sum_{i,j=1}^n g^i(x)h^j(x).$$

Using the properties of $X_p$ mentioned above, we have:

$$\begin{aligned} X_p f &= X_p\left(\partial_i f(p)(x^i - p^i)\right) \\ &= \partial_i f(p) X_p(x^i - p^i) \\ &= \partial_i f(p) v^i \\ &= D_{v_p} f \end{aligned}$$

This shows that $\Phi_p$ is onto. Hence, $\mathbf{T}_p\mathbb{R}^n \cong \mathrm{Der}_p(\mathbb{R}^n)$. Since a basis for $T_p\mathbb{R}^n$ is given by $\{(e_i)_p\}_{i=1}^n$, where $e_i$ is the $i$-th coordinate vector, it is clear that a basis for $\mathrm{Der}_p(\mathbb{R}^n)$ is given by $\{D_{(e_i)_p}\}_{i=1}^n$. Written down more explicitly, the basis is given by

$$\left.\frac{\partial}{\partial x_i}\right|_p : f \mapsto \frac{\partial f}{\partial x_i}(p)$$

for $i = 1, \ldots, n$

- Let $A = \mathrm{C}(\mathbb{R}^n)$. We show that that no nonzero derivations of $A$ over $\varepsilon$ exist. Let $f \in \mathrm{C}(\mathbb{R}^n)$ and let $D$ be a derivation. Since $D(c) = 0$, we can WLOG assume that $f(p) = 0$. Otherwise, simply consider $f - f(p)$. Moreover, any such $f$ can be written as a sum of non-negative functions,

$$f = f^+ - f^- \qquad f^+ = \max\{f(x), 0\} \quad f^- = \max\{-f(x), 0\}$$

Since, $D(f) = D(f^+) - D^(f^-)$, it suffices to show that $D(f) = 0$ for any non-negative $f \in \mathrm{C}(\mathbb{R}^n)$ with $f(p) = 0$.

Since $f$ is non-negative, $f = g^2$ where $g = \sqrt{f}$ is a non-negative function in $\mathrm{C}(\mathbb{R}^n)$. Note that

$$D(f) = D(g^2) = \sqrt{f(p)}D(g) + D(g)\sqrt{f(p)} = 0.$$

Hence, $D \equiv 0$.

Here are some remarks about the problem:

(1) When $f \in \mathrm{C}^\infty(\mathbb{R}^n)$, our argument in the first part crucially used the smoothness of $f$ since we expanded the function in a Taylor series.

(2) The argument in the second part breaks down for at least two reasons when $f \in \mathrm{C}^\infty(\mathbb{R}^n)$:

   (a) $f^+$ and $f^-$ might not be smooth and thus the argument does not work for $f \in \mathrm{C}^\infty(\mathbb{R}^n)$.

   (b) The square root of a non-negative smooth function, $f$, might not be a non-negative smooth function.

(3) It is certainly a bit surprising that the space of derivations over the algebra of continuous functions is zero. In any case, this problem highlights

that defining the notion of a *tangent space* and *dimension* is a tricky business if we intend to work with topological manifolds.

This completes the proof.

**Exercise 4.** The solution is given below:

- Let $F \in \mathsf{Diff}(X)$. Define the following action:

  $$\theta : \mathsf{Diff}(X) \times \mathsf{T}^*X \to \mathsf{T}^*X \qquad \theta(F,(p,\omega_p)) = (F(p),(dF^{-1}_{F(p)})^*(\omega_p))$$

  Here $(dF^{-1}_{F(p)})^*(\omega_p)$ is simply the pullback of $\omega_p \in T^*_pX$ by the linear map $(dF^{-1}_{F(p)})^*$. For fixed $F \in \mathsf{Diff}(X)$, we get a map, $\theta_F : \mathsf{T}^*X \to \mathsf{T}^*X$ which can be visualized in the diagram below

  $$\begin{tikzcd} X \arrow[r, "F^{-1}"] \arrow[d, dotted, "\theta_F(\omega)"'] & X \arrow[d, "\omega"] \\ \mathsf{T}^*X & \mathsf{T}^*X \arrow[l, "(dF^{-1})^*"] \end{tikzcd} \qquad \begin{tikzcd} F(p) \arrow[r, mapsto] \arrow[d, dotted, mapsto] & p \arrow[d, mapsto] \\ (F(p),(dF^{-1}_{F(p)})^*(\omega_p)) & (p,\omega_p) \arrow[l, mapsto] \end{tikzcd}$$

  Let's check that $\theta$ indeed defines a group action. Clearly,

  $$\theta(1_X,(p,\omega_p)) = (p,\omega_p)$$

  Moreover, note that

  $$\begin{aligned} (d(G\circ F)^{-1}_{G\circ F(p)})^*(\omega_p) &= (d(F^{-1}\circ G^{-1})_{G\circ F(p)})^*(\omega_p) \\ &= (dF^{-1}_{F(p)} \circ dG^{-1}_{G\circ F(p)})^*(\omega_p) \\ &= (dG^{-1}_{G(F(p))})^*((dF^{-1}_{F(p)})^*(\omega_p)) \end{aligned}$$

  Therefore,

  $$\theta(G\circ F,(p,\omega_p)) = \theta(G,(F(p),(dF^{-1}_{F(p)})^*(\omega_p)))$$

  Hence, $\theta$ indeed defines a group action. Let $0_X$ be the zero section on of $\mathsf{T}^*X$. For each $F \in \mathsf{Diff}(X)$ and $p \in X$, we have that

  $$\theta(F,(p,(0_X)_p)) = (F(p),(dF^{-1}_{F(p)})^*((0_X)_p))$$

  Since $(dF^{-1}_{F(p)})^*$ is a linear map for each $p \in M$, it is clear that

  $$\theta(F,(p,(0_X)_p)) = (F(p),(0_X)_{F(p)})$$

  for each $p \in M$. Hence, $\theta_F(0_X) = 0_X$ for each $F \in \mathsf{Diff}(X)$.

- Recall that a $(q,\omega_q) \in \mathsf{T}^*X$ is such that $\omega_q \in \mathsf{T}^*_qX$ for some $q \in X$. We show that there is a natural 1-form, $\alpha$, on $\mathsf{T}^*X$ called the *tautological 1-form*. Consider the natural projection map,

  $$\pi : \mathsf{T}^*X \to X \qquad \pi((q,\omega_q)) = q$$

Taking pullback yields a map

$$d\pi^*_{(q,\omega_q)} : \mathsf{T}^*_q X \to \mathsf{T}^*_{\omega_q}(\mathsf{T}^* X)$$

The *tautological 1-form*, $\alpha$, on $\mathsf{T}^*X$ is defined as

$$\alpha_{(q,\omega_q)} = d\pi^*_{(q,\omega_q)}(\omega_q)$$

In other words, if $v \in \mathsf{T}_{(q,\omega_q)}(\mathsf{T}^*X)$, then

$$\alpha_{(q,\omega_q)}(v) = \omega(d\pi_{(q,\omega_q)}(v))$$

Let $F \in \mathsf{Diff}(X)$. Let $dF^*$ denote the induced smooth map

$$dF^* : \mathsf{T}^*X \to \mathsf{T}^*X$$

$dF^*$ is defined by the rule

$$dF^*(p,\gamma_p) = (F(p),(dF^{-1}_{F(p)})^*(\gamma_p)) \qquad (p,\gamma_p) \in \mathsf{T}^*X$$

In what follows, denote $G = dF^*$. We claim that $\alpha$ is invariant under $G$. That is,

$$G^*(\alpha) = \alpha$$

That is, for each $(p,\gamma_p) \in \mathsf{T}^*X$, we have that

$$dG^*_{(p,\gamma_p)}(\alpha_{G(p,\gamma_p)}) = \alpha_{(p,\gamma_p)}$$

Before moving on, note that the following diagram commutes

$$\begin{tikzcd}
\mathsf{T}^*X \arrow[r, "G"] \arrow[d, "\pi"] & \mathsf{T}^*X \arrow[d, "\pi"] \\
X \arrow[r, "F"] & X
\end{tikzcd}$$

Let $G(p,\gamma) = (p',\gamma'_{p'})$ We have the following:

$$\begin{aligned}
dG^*_{(p,\gamma)}(\alpha_{G(p,\gamma)}) &= dG^*_{(p,\gamma)}(d\pi^*_{(p',\gamma')}(\gamma'_{p'})) \\
&= d(\pi \circ G)^*_{(p',\gamma')}(\gamma'_{p'}) \\
&= d(F \circ \pi)^*_{(p',\gamma')}(\gamma'_{p'}) \\
&= d\pi^*_{(p,\gamma)}(dF^*_{p'}(\gamma'_{p'})) \\
&= d\pi^*_{(p,\gamma)}(\gamma_p) \\
&= \alpha_{(p,\gamma)}
\end{aligned}$$

This proves the claim.

•

- We compute the exterior derivative in coordinates. For $q \in X$, let $(U, (x^1, \cdots, x^n))$ be a chart in $X$ containing $q$. Let $(\widetilde{U}, (x^1, \cdots, x^n, p^1, \cdots, p^n))$ be the corresponding chart in $\mathsf{T}^*X$ containing $(q, \omega_q)$ such that

$$\omega = \omega_i dx^i$$

is the coordinate representation of $\omega$. Write some $v \in T_{(q,\omega_q)}(\mathsf{T}^*X)$ as

$$v = v^i \frac{\partial}{\partial x^i} + v^k \frac{\partial}{\partial p^k}$$

Since $d\pi$ is essentially the block matrix

$$d\pi = \begin{pmatrix} 1_{n\times n} & 0_{n\times n}, \end{pmatrix}$$

we have:

$$\begin{aligned} \alpha_{(q,\omega_q)}(v) &= \omega(d\pi_{(q,\omega_q)}(v)) \\ &= \omega\left(v^i \frac{\partial}{\partial x^i}\right) \\ &= \omega_j dx^j\left(v^i \frac{\partial}{\partial x^i}\right) \\ &= \omega_j v^i dx^j\left(\frac{\partial}{\partial x^i}\right) \\ &= \omega_i v^i \end{aligned}$$

In other words,

$$\alpha_{(q,\omega_q)} = d\pi^*_{(q,\omega_q)}(\omega) = \omega_i dx^i$$

Therefore,

$$d\alpha_{(q,\omega_q)} = d\omega_i \wedge dx^i$$

Clearly, $d\alpha$ is closed since

$$d(d\omega_i \wedge ddx^i) = d^2\omega_i \wedge ddx^i - d\omega_i \wedge d^2dx^i$$

because $d^2 = 0$. Moreover, $d\alpha$ is non-degenerate. For some $X \in T(\mathsf{T}^*X)$ (a vector field in $\mathsf{T}^*X$), assume that $d\alpha(X, Y) = 0$ for each $Y \in T(\mathsf{T}^*X)$. In the local coordinates chosen as above, write $X$ as

$$a^i \frac{\partial}{\partial x^i} + b^k \frac{\partial}{\partial p^k}$$

We have

$$0 = d\alpha_{(q,\omega_q)}\left(X^i \frac{\partial}{\partial x^i} + X^k \frac{\partial}{\partial p^k}, \frac{\partial}{\partial x^j}\right) = -a^j$$

$$0 = d\alpha_{(q,\omega_q)}\left(X^i \frac{\partial}{\partial x^i} + X^k \frac{\partial}{\partial p^k}, \frac{\partial}{\partial p^j}\right) = b^j$$

Hence, $X = 0$ in each local coordinate chart. Therefore, $X \equiv 0$. Hence, $d\alpha$ is non-degenerate.

**Exercise 5.** The solution is given below:

- (1) $\mathsf{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ which has dimension $n^2$. An open subset of a manifold is a manifold whose dimension is the same as that of the ambient manifold. Hence,

$$\dim \mathsf{GL}(n, \mathbb{R}) = n^2$$

  (2) Consider the map

$$\det : \mathsf{GL}(n, \mathbb{R}) \to \mathbb{R}^* \qquad A \mapsto \det A$$

  Clearly, $\mathsf{SL}(n, \mathbb{R}) = \det^{-1}(1)$. We claim that det has constant rank 1. It suffices to show that this at $I_n \in \mathsf{GL}(n, \mathbb{R})$ since det is Lie group homomorphism. Let $X \in T_{I_n}\mathsf{GL}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ and consider the curve $\gamma(t) = I_n + tX$ in $\mathbb{R}^{n^2}$[3] We compute

$$\left.\frac{d}{dt}\right|_{t=0} \det(I + tX)$$

  Note that to first order:

$$\begin{aligned}
\det(I + tX) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot (I + tX)_{1,\sigma(1)} \cdot (I + tX)_{2,\sigma(2)} \cdot \dots \cdot (I + tX)_{n,\sigma(n)} \\
&= \prod_{i=1}^{n} (1 + tX_{ii}) + O(t^2) \\
&= 1 + t \sum_{i=1}^{n} X_{ii} + O(t^2)
\end{aligned}$$

  Therefore,

$$\left.\frac{d}{dt}\right|_{t=0} \det(I + tX) = \sum_{i=1}^{n} X_{ii} = \operatorname{Tr} X$$

[3]For small enough $t$, $\gamma(t)$ is contained in $\mathsf{GL}(n, \mathbb{R})$ since $\mathsf{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ so the map is well-defined for small enough $t$.

Clearly, the linear map $X \mapsto \operatorname{Tr} X$ is surjective. Hence, $dF_{I_n}$ has rank 1. By the regular level set theorem, we have

$$\dim \mathbf{SL}(n, \mathbb{R}) = \mathbf{GL}(n, \mathbb{R}) - \operatorname{rank} \det = n^2 - 1.$$

(3) Consider

$$\Phi : \mathbf{GL}(n, \mathbb{R}) \to \mathbf{GL}(n, \mathbb{R}) \qquad A \mapsto A^T A$$

It can be shown that $\Phi$ has constant rank[4]. Therefore, by the regular level set theorem, we have

$$\dim \mathbf{O}(n, \mathbb{R}) = \mathbf{GL}(n, \mathbb{R}) - \operatorname{rank} d\Phi_{I_n}$$

We compute $\operatorname{rank} d\Phi_{I_n}$. The tangent space of $\mathbf{GL}(n, \mathbb{R})$ isomorphic to $\mathbf{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Fix any $A \in \mathbf{M}_n(\mathbb{R})$. For any small enough $\varepsilon > 0$, consider a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{O}(n, \mathbb{R})$ such that $\gamma(0) = I_n$ and $\gamma'(0) = A$[5]. We have:

$$\begin{aligned} d\Phi_{I_n}(A) &= (\Phi \circ \gamma)'(0) \\ &= \frac{d}{dt}\gamma(t)^T \gamma(t)\bigg|_{t=0} \\ &= \gamma'(0)^T \gamma(0) + \gamma(0)^T \gamma'(0) \\ &= A + A^T \end{aligned}$$

Since $A + A^T$ is symmetric, the image of $d\Phi_{I_n}$ is contained in the vector space of $n$-by-$n$ symmetric matrices. In fact, it is equal to this vector space. This is because for any

$$d\Phi_{I_n}(B) = \frac{B + B^T}{2} = B$$

for any $n$-by-$n$ symmetric matrix, $B$. Therefore,

$$\dim \mathbf{O}(n, \mathbb{R}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

- Each group is a subset of $\mathbb{R}^{n^2}$. Therefore, it will be compact if and only if it is closed an bounded with respect to the Euclidean topology. Since all norms on $\mathbb{R}^{n^2}$ are equivalent, we choose to work with the infinity norm, $\|\cdot\|_\infty$.

[4]Use the equivariant rank theorem which is a generalization of the result that a Lie group homomorphism has constant rank.
[5]Such curves are guaranteed to exist!

(1) Consider the sequence of matrices:

$$A_m = mI_n$$

Clearly, $\|A_m\|_\infty = m$. Therefore, $\mathsf{GL}(n, \mathbb{R})$ is not a bounded set since it contains matrices of arbitrarily large norm.

(2) Consider the sequence of matrices:

$$A_m = \begin{bmatrix} m & 0 & \cdots & 0 \\ 0 & \frac{1}{m} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

By construction, $\det A_m = 1$ and $\|A_m\|_\infty = m$. Therefore, $\mathsf{SL}(n, \mathbb{R})$ is not a bounded set since it contains matrices of arbitrarily large norm.

(3) Consider the function:

$$\Phi : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \qquad \Phi(A) = A^T A$$

Clearly, $\mathsf{O}(n, \mathbb{R}) = \Phi^{-1}(I_n)$. Therefore, $\mathsf{O}(n, \mathbb{R})$ is a closed set. Each entry in a matrix $O \in \mathsf{O}(n, \mathbb{R})$ is bounded in absolute value by 1 since each matrix in $O \in \mathsf{O}(n, \mathbb{R})$ has orthonormal columns. Therefore, $\|O\|_\infty \leq 1$ for every $O \in \mathsf{O}(n, \mathbb{R})$. As a result, $O \in \mathsf{O}(n, \mathbb{R})$ is compact.

- $\mathsf{O}(n, \mathbb{R})$ is not connected. If not, then $\mathsf{O}(n, \mathbb{R})$ would be path connected since $\mathsf{O}(n, \mathbb{R})$ is a smooth manifold. However, $+I_n$ and $-I_n$ cannot be connected by a continuous path by the continuity of the determinant function.

We show every $A \in \mathsf{SO}(n, \mathbb{R})$ can be connected to $I_n$. First, we argue that given any two unit vectors $v, w \in \mathbb{R}^n$, there is a path $\gamma(t) \in \mathsf{SO}(n, \mathbb{R})$ such that:

$$\gamma(0)v = v$$
$$\gamma(1)v = w$$

That is, any two unit vectors in $\mathbb{R}^n$ can be *continuously rotated*. Choose a $u \in \mathbb{R}^n$ as follows:

(1) If $v$ and $w$ are linearly independent, apply the Gram-Schmidt algorithm and choose $u$ such that $u \perp v$ and $u \in \operatorname{span}\{v, w\}$.

(2) If $v$ and $w$ are linearly dependent ($w = -v$), then take $u$ to be any unit vector in $v^\perp$.

Let $V = \text{span}\{v, u\}$. One can then consider a one-parameter family of rotations, $R_\phi \in \mathbf{SO}(2, \mathbb{R})$ that act on $V$. Since $w \in V$, there is an angle $\phi_0$ such that (in the above constructed basis):

$$w = \begin{bmatrix} R_{\phi_0} & 0 \\ 0 & I_{n-2} \end{bmatrix} v.$$

Define the path

$$\gamma(t) := \begin{bmatrix} R_{t\phi_0} & 0 \\ 0 & I_{n-2} \end{bmatrix}$$

The image of $\gamma$ is clearly contained in $\mathbf{SO}(n, \mathbb{R})$ and is such that

$$\gamma(0) = R(0)v = v$$
$$\gamma(1) = R(1)v = w$$

Any $A \in \mathbf{SO}(n, \mathbb{R})$ is represented by an orthonormal basis $(a_1, \dots, a_n)$ over vectors in $\mathbb{R}^n$. Apply the above procedure recursively: find a path $\gamma_1(t) \in \mathbf{SO}(n, \mathbb{R})$ such that

$$\gamma_1(t)a_1 = e_1$$

Then choose a path $\gamma_2$ taking $\gamma_1(1)a_2$ to $e_2$. Note that any such $\gamma_2$ leaves $e_1$ invariant. Indeed $e_1 \perp e_2, \gamma_1(1)a_2$ [6]. So, $e_1$ is in the complement of the subspace in which the rotation happens and is thus left invariant. Proceed recursively now and consider the paths $\gamma_1(t), \cdots, \gamma_n(t)$. Consider

$$\gamma = \gamma_n \circ \cdots \circ \gamma_1$$

Based on the above remarks, it is clear that

$$\gamma(0)a_i = a_i$$
$$\gamma(1)a_i = e_i$$

for $i = 1, \cdots, n$. Hence, $\mathbf{SO}(n, \mathbb{R})$ is path-connected and hence connected since $\mathbf{SO}(n, \mathbb{R})$ is a smooth manifold.

- (1) Consider the function:

  $$\det : \mathbb{R}^{n^2} \to \mathbb{R}$$

  Clearly, $\mathbf{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$. Therefore, $\mathbf{GL}(n, \mathbb{R})$ is not connected. Otherwise, $\mathbb{R} \setminus \{0\}$, which is the image of the det map, will be connected. Hence, $\mathbf{GL}(n, \mathbb{R})$ has at least two components. We show that it has exactly 2 components. We have:

  $$\mathbf{GL}(n, \mathbb{R}) = \mathbf{GL}_n^+(\mathbb{R}) \bigsqcup \mathbf{GL}_n^-(\mathbb{R}),$$

[6]Applying $\gamma_1$ to an orthonormal basis results in an orthonormal basis

where $\mathsf{GL}^{\pm}(n,\mathbb{R})$ is the set of all elements in $\mathsf{GL}(n,\mathbb{R})$ with positive/negative determinant. It suffices to show that $\mathsf{GL}^{+}(n,\mathbb{R})$ is path-connected since $\mathsf{GL}^{-}(n,\mathbb{R})$ is diffeomorphic to $\mathsf{GL}^{+}(n,\mathbb{R})$.

We use the singular value decomposition. Let

$$A = U\Sigma V$$

be the singular value decomposition of $A$. Here $U$ and $V$ are unitary matrices and $\Sigma$ is a diagonal matrix consisting of the singular values of $A$ which are all non-negative[7]. Since $A$ has positive determinant, the singular values of $A$ are all positive real numbers.

Since $\det A > 0$, $\det U = \det V$. Therefore, both $U$ and $V$ are in the same component of $\mathsf{O}(n,\mathbb{R})$. WLOG, assume that both matrices are contained in $\mathsf{SO}(n,\mathbb{R})$. Since $\mathsf{SO}(n,\mathbb{R})$ is connected, there exist paths $\gamma_1(t)$ and $\gamma_2(t)$ in $\mathsf{SO}(n,\mathbb{R})$ such that

$$\gamma_1(0) = U \quad \gamma_1(1) = I_n$$

$$\gamma_1(0) = V \quad \gamma_1(1) = I_n$$

Consider the path

$$\gamma(t) = \gamma_1(t)\Sigma\gamma_2(t)$$

Clearly, $\gamma(t)$ is in $\mathsf{SO}(n,\mathbb{R})$ such that

$$\gamma_1(0) = A \quad \gamma_1(1) = \Sigma$$

Since $\Gamma$ has positive entries, there exists a smooth curve $\beta$ such that $\beta(s) \in \mathsf{SO}(n,\mathbb{R})$ and that

$$\beta_1(0) = \Sigma \quad \beta(1) = I_n$$

Simply consider $\beta \circ \gamma$. This shows that $\mathsf{GL}^{+}(n,\mathbb{R})$

(2) Consider the continuous surjective map

$$\Phi\colon \mathsf{GL}^{+}(n,\mathbb{R}) \to \mathsf{SL}(n,\mathbb{R}) \qquad A \mapsto \frac{A}{(\det A)^{\frac{1}{n}}} \in \mathsf{SL}(n,\mathbb{R}).$$

Since $\mathsf{GL}^{+}(n,\mathbb{R})$ is connected, $\mathsf{SL}(n,\mathbb{R})$ since it is the image of a connected set under a continuous map.

[7]This is crucial in this proof.

- We first pin down structure of the matrices in $\mathsf{SO}(2,\mathbb{R})$ and $\mathsf{SO}(3,\mathbb{R})$. We claim that every element in $A \in \mathsf{SO}(2,\mathbb{R})$ can be written as:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Indeed, the columns of a matrix in $\mathsf{SO}(2,\mathbb{R})$ are orthonormal. Hence, the first column must be a unit vector in $\mathbb{R}^2$. Hence, it can be written as the unit vector shown in the first column in the matrix above. It is clear that the only unit vector orthogonal to the first column vector is the second column vector shown in the matrix above. Clearly, every matrix in $\mathsf{SO}(2,\mathbb{R})$ is a rotation.

We now claim that every element in $A \in \mathsf{SO}(3,\mathbb{R})$ is conjugate to a matrix of the form:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here, we use the easily proved fact that every (real or complex) eigenvalue of a matrix in $\mathsf{SO}(n,\mathbb{R})$ has absolute value 1.

The characteristic polynomial of $A$ is a degree three polynomial so it must have a real root. If it has three real roots, then the eigenvalues are $(1,1,1)$ or $(-1,-1,1)$. Hence, $A$ is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is of the form above. If it has only one root, then it must be 1 and the complex eigenvalues come in conjugate pairs. Let $V$ denote the eigenspace of eigenvalue 1 and let $V^{\perp}$[8] be the eigensapce corresponding to the complex eigenvalues. Let the complex eigenvalues be $\lambda = \cos\theta + i\sin\theta =$ for some $\theta$ and $\overline{\lambda}$ along with complex eigenvectors $v$ and $\overline{v}$ respectively. It is quite easy to see that

$$V^{\perp} = \operatorname{Span}\{v+\overline{v}, i(v-\overline{v})\}$$

invariant under the action of $A$:

$$A(v+\overline{v}) = \cos\theta(v+\overline{v}) + \sin\theta i(v-\overline{v})$$
$$Ai(v-\overline{v}) = -\sin\theta(v+\overline{v}) + \cos\theta i(v-\overline{v})$$

[8]The matrix is orthogonally diagonalizable!

Hence, $A$ restricted to $V^\perp$ is of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This proves the claim. Hence, every element in $\mathbf{SO}(3, \mathbb{R})$ fixes a vector, $v$, and maps the plane orthogonal to $v$ via a rotation.

For $n \geq 4$, consider the following matrix:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi & 0 \\ 0 & 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & I_{n-4} \end{pmatrix}$$

where $\theta \in \mathbb{Q}/\mathbb{Z}$, the multiplicative group of roots of unity, and $\phi \in \mathbb{S}^1 \setminus \mathbb{Q}/\mathbb{Z}$. This is not a rotation.

- Recall that every matrix $A \in \mathbf{SO}(2, \mathbb{R})$ can be written as:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Consider $\mathbb{S}^1 \subseteq \mathbb{C}$. Define:

$$F : \mathbb{S}^1 \to \mathbf{SO}(2, \mathbb{R}) \qquad e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$F$ is smooth because $F$ can be thought of as the restriction of a smooth map

$$\tilde{F} : \mathbb{C} \to M_2(\mathbb{R}) \cong \mathbb{R}^4 \qquad re^{i\theta} \mapsto \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$$

Since $\mathbb{S}^1 \subseteq \mathbb{C}$ and $\mathbf{SO}(2, \mathbb{R}) \subseteq M_2(\mathbb{R})$ are embedded submanifolds of $M_2(\mathbb{R})$, the restriction map is guaranteed to smooth. Consider

$$\widetilde{G} : M_2(\mathbb{R}) \to \mathbb{C} \qquad \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto a + ib$$

Clearly, $\widetilde{G}$ is a smooth map and its restriction

$$G : \mathbf{SO}(2, \mathbb{R}) \to \mathbb{S}^1 \qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto \cos\theta + i\sin\theta = e^{i\theta}$$

is also smooth as argued above. Clearly, $F$ and $G$ are inverse of each other. Hence,

$$\mathbf{SO}(2, \mathbb{R}) \cong \mathbb{S}^1$$

Let $B^3_\pi$ be a 3-ball of radius $\pi$. Consider the equivalence relation on $\partial B^3_\pi =$ by $x \sim y$ if $x = -y$. It is a standard fact that

$$\mathbb{RP}^3 \cong B^3_\pi / \partial B^3_\pi$$

From the previous part, we know that an element in $\mathsf{SO}(3, \mathbb{R})$ can be specified by a some $v \in \mathbb{R}^3$ and a rotation in $v^\perp$ by an angle $\theta \in [-\pi, \pi]$ radians. A rotation by $\pi$ is equivalent to a rotation by $-\pi$. Therefore, we can represent any element of $\mathsf{SO}(3, \mathbb{R})$, besides the identity by the ordered pair $(v, \theta)$, where $v$ is the unit vector in the direction of the rotation, and $\theta$ is the magnitude between $-\pi$ and $\pi$. Note that $\theta v \in B^3_\pi$. An explicit diffeomorphism between $\mathsf{SO}(3, \mathbb{R})$ and $B^3_\pi / \partial B^3_\pi$ is

$$f(v, \theta) = [\theta v],$$

and

$$f(I) = 0.$$

It is a simple matter to check that this is a well-defined diffeomorphism. Hence,

$$\mathsf{SO}(3, \mathbb{R}) \cong B^3_\pi / \partial B^3_\pi \cong \mathbb{RP}^3$$

# MATH 740 HOMEWORK 2

W. GOLDMAN

(Due 14 March 2024)

A good general reference is *Introduction to Smooth Manifolds,* by John M. Lee, ISBN 978-1-4899-9475-2, Springer Graduate Texts in Mathematics 218, Second Edition (2012).

**Problem 6.**

(1) Consider the vector field
$$Y := e^x \frac{\partial}{\partial y}$$
on the plane $M = \mathbb{R}^2$. Integrate $X$ to obtain a *local* flow $\varphi_t(x, y)$ (where $t$ is defined in some neighborhood of 0, which may vary with the initial condition $(x, y)$.[1]

(2) Is $Y$ *complete*?, that is, is $\varphi_t$ defined for all time?

(3) Show that the diffeomorphism
$$\mathbb{R}^2 \xrightarrow{\Phi} \mathbb{R}^2$$
$$(x, y) \longmapsto (x, e^{-x} y)$$
satisfies $\Phi_* Y = \frac{\partial}{\partial y}$.

(4) Verify directly that
$$F\big(\varphi_t(x, y)\big) = \tau_t F(x, y)$$
where $\tau_t(x, y) = (x, y + t)$ is the one-parameter group of translations generated by the vector field $\frac{\partial}{\partial y}$.

(5) Deduce this from uniqueness of flows generated by vector fields.

(6) Show that $\varphi_t$ preserves Euclidean area.

(7) Deduce this by computing the interior product $\iota_Y(\omega)$, its exterior derivative $d\iota_Y(\omega)$, and the Lie derivative $\mathcal{L}_Y(\omega)$ where $\omega = dx \wedge dy$ is the Euclidean area form, using Cartan's formula
$$\mathcal{L}_Y = d\iota_Y + \iota_Y d.$$

*Date*: March 6, 2024.

[1]The condition that $\varphi$ is a local flow is that $\varphi_0(x, y) = (x, y)$ and $\varphi_s \circ \varphi_t = \varphi_{s+t}$ whenever $(\varphi_s \circ \varphi_t)(x, y)$ and $\varphi_{s+t}(x, y)$ are defined.

(8) Let $X := e^y \frac{\partial}{\partial x}$. Find a diffeomorphism $\Psi$ such that
$$\Psi_*(X) = \frac{\partial}{\partial x}.$$
(9) Compute the Lie bracket $[X, Y]$. Use this to deduce that no diffeomorphsm $\Psi$ exists such that
$$\Psi_*(Y) = \frac{\partial}{\partial y} \text{ and } \Psi_*(X) = \frac{\partial}{\partial x}.$$
(10) Show $X+Y$ is incomplete, although both $X$ and $Y$ are complete.

**Problem 7.**
Let $a, b \in \mathbb{R}$ with $a < b$ and $(a, b) \xrightarrow{\gamma} \mathbb{E}^2$ a regular plane curve parametrized by arc length. The matrix
$$\mathbb{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
represents counter-clockwise orthogonal rotation. Show that
$$\mathsf{T}(s) := \gamma'(s)$$
$$\mathsf{N}(s) := \mathbb{J}\,\gamma'(s)$$
defines a positively oriented orthonormal frame field along $\gamma$. Define the *signed curvature* $\mathsf{k}^{\pm}(s)$ of $\gamma$ by:
$$\mathsf{T}'(s) = \mathsf{k}^{\pm}(s)\mathsf{T}(s).$$
Prove that:

(1) If $g \in \mathsf{Isom}^+(\mathbb{E}^2)$, then the signed curvature the transformed curve $g \circ \gamma$ also has signed curvature $\mathsf{k}^{\pm}(s)$.
(2) Conversely suppose that two regular plane curves $\gamma_1, \gamma_2$ have the same signed curvature function. Show there exists a unique $g \in \mathsf{Isom}^+(\mathbb{E}^2)$ such that $\gamma_2 = g \circ \gamma_1$.
(3) Let $(a, b) \xrightarrow{f} \mathbb{R}$ be a smooth function. Let $p_0 \in \mathbb{E}^2$ and $(\mathsf{T}_0, \mathsf{N}_0)$ be a positively oriented orthonormal frame. Show there is a unique regular plane curve $\gamma(s)$ with:
   (a) The signed curvature $\mathsf{k}^{\pm}(s)$ of $\gamma$ equals $f$;
   (b) At time $s = 0$, the unit tangent and normal vector fields equal $\mathsf{T}_0, \mathsf{N}_0$.

**Problem 8.**

(1) Let $(a,b) \xrightarrow{\gamma} \mathbb{E}^3$ a regular space curve parametrized by arc length. That is, the *velocity* $\gamma'(s)$ has unit length, and equals the *unit tangent vector field* $\mathsf{T}(s)$. Assume the *acceleration*

$$\frac{D}{ds}\gamma'(s) = \frac{D}{ds}\mathsf{T}(s) := \mathsf{T}'(s) = \gamma''(s)$$

is nonzero for all $s \in (a,b)$ (so that the unit normal vector field $\mathsf{N}(s)$ can be uniquely defined).

(2) Prove that the acceleration $\gamma''(s)$ is orthogonal $\mathsf{T}(s)$ and define the *unit normal vector field* $\mathsf{N}(s)$ by:

$$\mathsf{N}(s) := \frac{1}{\|\gamma''(s)\|}\gamma''(s)$$

and the *curvature* by

$$\mathsf{k}(s) := \|\gamma''(s)\|.$$

Define the *binormal vector field* by:

$$\mathsf{B}(s) := \mathsf{T}(s) \times \mathsf{N}(s)$$

and prove that $\|\mathsf{B}(s)\| = 1$.

(3) Prove that for each $s$, the ordered triple $\big(\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s)\big)$ is a positively oriented orthonormal basis of $\mathbb{R}^3$ (that is, a positively oriented orthonormal *frame*).

(4) Define the *torsion* $\boldsymbol{\tau}(s)$ of $\gamma$ as the $\mathsf{B}(s)$-component of the derivative $\mathsf{N}'(s)$:

$$\boldsymbol{\tau}(s) := \mathsf{N}'(s) \cdot \mathsf{B}(s).$$

Show that the *moving frame*[2] $\big(\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s)\big)$ satisfies the Frenet-Serret *structure equations:*

$$(*) \qquad \begin{bmatrix} \mathsf{T}'(s) \\ \mathsf{N}'(s) \\ \mathsf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{k}(s) & 0 \\ -\mathsf{k}(s) & 0 & \boldsymbol{\tau}(s) \\ 0 & -\boldsymbol{\tau}(s) & 0 \end{bmatrix} \begin{bmatrix} \mathsf{T}(s) \\ \mathsf{N}(s) \\ \mathsf{B}(s) \end{bmatrix}.$$

(5) Show that $\mathsf{k}(s)$ and $\boldsymbol{\tau}(s)$ are invariant under the group $\mathsf{Isom}^+(\mathbb{E}^3)$, that is, if we replace the curve $\gamma$ by the composition

$$\widetilde{\gamma} := g \circ \gamma,$$

where $g \in \mathsf{Isom}^+(\mathbb{E}^3)$ is an orientation-preserving isometry, then the corresponding curvature and torsion functions $\widetilde{\mathsf{k}}(s), \widetilde{\boldsymbol{\tau}}(s)$ are equal:

$$\widetilde{\mathsf{k}}(s) = \mathsf{k}(s), \quad \widetilde{\boldsymbol{\tau}}(s) = \boldsymbol{\tau}(s).$$

[2]In French, *repère mobile.*

(6) Show any pair of smooth functions $\mathsf{k}(s), \boldsymbol{\tau}(s)$, where $\mathsf{k}(s) > 0$, can be realized by a smooth regular space curve $\gamma(s)$ and this curve is unique up to composition with $\mathsf{Isom}^+(\mathbb{E}^3)$.

(Hint: set this up as initial value problem. The Frenet-Serret frame field $\big(\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s)\big)$ and the curvature and torsion are defined by the structure equations (*). Consider a fiber bundle over $(a, b)$ with total space

$$\mathcal{E} := (a, b) \times \mathbb{E}^3 \times \mathbf{F_3}$$

where $\mathbf{F_3}$ is the set of positively oriented orthonormal frames[3] in $\mathbb{R}^3$.

This makes $\mathcal{E} \longrightarrow (a, b)$ into a *principal* $\mathsf{Isom}^+(\mathbb{E}^3)$*-bundle over the parameter space* $(a, b)$, and we seek a *section* of this bundle (which will be a solution of (*)). Write the coordinates of $\mathcal{E}$ as

$$\Big(s; (x, y, z); (t_1, t_2, t_3); (n_1, n_2, n_3); (b_1, b_2, b_3)\Big)$$

where:

- $s \in (a, b)$ corresponds to "time" (or "arc length");
- $(x, y, z)$ are the coordinates of the space curve $\gamma$;
- $(t_1, t_2, t_3)$ are the components of the unit tangent vector field $\mathsf{T}$;
- $(n_1, n_2, n_3)$ are the components of the unit normal vector field $\mathsf{N}$;
- $(b_1, b_2, b_3)$ are the components of the unit binormal vector field $\mathsf{B}$.

Then (*) corresponds to the vector field $\xi$ on $\mathcal{E}$ defined by:

$$\begin{aligned}\xi := \partial_s + (t_1\partial_x + t_2\partial_y + t_3\partial_z) \\ + \mathsf{k}(s)(n_1\partial_{t_1} + n_2\partial_{t_2} + n_3\partial_{t_3}) \\ + \big( - \mathsf{k}(s)t_1 + \boldsymbol{\tau}(s)b_1\big)\partial_{n_1} \\ + \big( - \mathsf{k}(s)t_2 + \boldsymbol{\tau}(s)b_2\big)\partial_{n_2} \\ + \big( - \mathsf{k}(s)t_3 + \boldsymbol{\tau}(s)b_3\big)\partial_{n_3} \\ + \big( - \boldsymbol{\tau}(s)\big)(n_1\partial_{b_1} + n_2\partial_{b_2} + n_3\partial_{b_3}).\end{aligned}$$

Now solve an initial value problem.)

[3] $\mathbf{F_3}$ is an $\mathsf{SO}(3)$-*torsor*, that is, any pair of positively oriented frames are related by a *unique* element of $\mathsf{SO}(3)$.

(7) Explain why the matrix

$$\begin{bmatrix} 0 & \mathsf{k}(s) & 0 \\ -\mathsf{k}(s) & 0 & \boldsymbol{\tau}(s) \\ 0 & -\boldsymbol{\tau}(s) & 0 \end{bmatrix}$$

appearing in (*) is skew-symmetric.

(8) Find all space curves of constant curvature and constant torsion.

## 2. Homework 2

**Exercise 6.** The solution is given below:

- Let $\gamma(t) = (x(t), y(t))$ be an integral curve for $X$. The associated system of ODE is
$$\frac{dx}{dt} = 0 \qquad \frac{dy}{dt} = e^{x(t)}$$
Consider the initial condition $\gamma(0) = (x_0, y_0)$. Since the system of ODE's is uncoupled, we first solve the first ODE and then solve the second ODE. The solution to the first ODE is clearly,
$$x(t) = x_0.$$
Plugging this expression in the second ODE, we get,
$$\frac{dy}{dt} = e^{x_0}, \qquad y(0) = y_0.$$
The solution to the second ODE is
$$y(t) = y_0 + e^{x_0}t$$
Hence, the corresponding integral curve is
$$\gamma(t) = (x_0, y_0 + e^{x_0}t)$$
The corresponding flow is
$$\varphi_t(x_0, y_0) = (x_0, y_0 + e^{x_0}t)$$
- $Y$ is complete since for each $(x_0, y_0) \in \mathbb{R}^2, \varphi_t(x_0, y_0)$ is defined for all $t \in \mathbb{R}$.
- Write the coordinates in the domain as $(u, v)$. The inverse of $\Phi$ is given by the formula:
$$\Phi^{-1}(u, v) = (u, e^u v)$$
The differential of $F$
$$d\Phi_{(x,y)} = \begin{bmatrix} 1 & 0 \\ -ye^{-x} & e^{-x} \end{bmatrix} \iff d\Phi_{\Phi^{-1}(u,v)} = \begin{bmatrix} 1 & 0 \\ -v & e^{-u} \end{bmatrix}$$
Moreover, have
$$Y_{\Phi^{-1}(u,v)} = e^u \frac{\partial}{\partial y}\bigg|_{F^{-1}(u,v)}$$
Therefore,
$$(\Phi_* Y)_{(u,v)} = d\Phi_{\Phi^{-1}(u,v)}(Y_{\Phi^{-1}(u,v)}) = \frac{\partial}{\partial v}\bigg|_{(u,v)}$$

- Note that

$$\begin{aligned}\Phi(\varphi_t(x_0, y_0)) &= \Phi(x_0, y_0 + e^{x_0}t) \\ &= (x_0, e^{-x_0}(y_0 + e^{x_0}t)) \\ &= (x_0, e^{-x_0}y_0 + t) = \tau_t(\Phi(x_0, y_0))\end{aligned}$$

- We have

$$\frac{d}{dt}\bigg|_{t=0} \Phi(\varphi_t(x_0, y_0)) = d\Phi_{(x_0,y_0)}\Big(\frac{d}{dt}\bigg|_{t=0} \varphi_t(x, y)\Big) \tag{1}$$

$$= d\Phi_{(x_0,y_0)}(Y_{(x_0,y_0)}) \tag{2}$$

$$= \frac{\partial}{\partial y}\bigg|_{\Phi(x_0,y_0)} \tag{3}$$

Since $\tau_t(x_0, y_0)$ is the flow generated by the vector field $\frac{\partial}{\partial y}$, by the uniqueness of the flow we have

$$\Phi(\phi_t(x_0, y_0)) = \tau_t(\Phi(x_0, y_0)) \tag{4}$$

- Fix some $t \in \mathbb{R}$ and consider

$$\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \varphi_t(x, y) = (x, y + e^x t)$$

The area form, $dA$, on $\mathbb{R}^2$ (the co-domain) is $dA_{(u,v)} = du \wedge dv$. Note that

$$\begin{aligned}\varphi_t^*(dA)_{(x,y)} &= dx \wedge d(y + e^x t) \\ &= dx \wedge d(y + e^x t dx) \\ &= dx \wedge dy + e^x t dx \wedge dx \\ &= dx \wedge dy \\ &= dA_{(x,y)}\end{aligned}$$

Hence,

$$\varphi_t^*(dA) = dA$$

- Note that

$$d(dx \wedge dy) = d^2x \wedge dy + dx \wedge d^2y = 0 + 0 = 0$$

If $Z$ is any vector field on $\mathbb{R}^2$, we have:

$$\iota_Y(\omega)(Z) = \omega(Y, Z) = \det\begin{bmatrix} dx(Y) & dx(Z) \\ dy(Y) & dy(Z)\end{bmatrix} = \det\begin{bmatrix} 0 & dx(Z) \\ e^x & dy(Z)\end{bmatrix} = -e^x dx(Z)$$

Therefore

$$\iota_Y(\omega) = -e^x dx$$

As a result

$$d(\iota_Y(\omega)) = -d(e^x dx) = -e^x dx \wedge dx = 0$$

Using Cartan's formula, we have:

$$\mathcal{L}_Y(\omega) = 0 + 0 = 0$$

This is sufficient to imply that $\omega$ is preserved under the flow generated by $Y$. See [Lee12, Proposition 12.37]

- Write the coordinates in the domain as $(u,v)$. The inverse of $\Psi$ is given by the formula:

$$\Psi^{-1}(u,v) = (ue^v, v)$$

The differential of $F$

$$d\Psi_{(x,y)} = \begin{bmatrix} e^{-y} & -xe^{-y} \\ 0 & 1 \end{bmatrix} \iff d\Phi_{\Phi^{-1}(u,v)} = \begin{bmatrix} e^{-v} & -u \\ 0 & 1 \end{bmatrix}$$

Moreover, have

$$Y_{\Phi^{-1}(u,v)} = e^v \left.\frac{\partial}{\partial x}\right|_{F^{-1}(u,v)}$$

Therefore,

$$(\Phi_* Y)_{(u,v)} = d\Phi_{\Phi^{-1}(u,v)}(Y_{\Phi^{-1}(u,v)}) = \left.\frac{\partial}{\partial u}\right|_{(u,v)}$$

- The Lie bracket is given by

$$\begin{aligned}[X,Y] &= \left[e^y \frac{\partial}{\partial x}, e^x \frac{\partial}{\partial y}\right] \\ &= X(e^x)\frac{\partial}{\partial y} - Y(e^y)\frac{\partial}{\partial x} \\ &= e^{(x+y)}\frac{\partial}{\partial y} - e^{(x+y)}\frac{\partial}{\partial x} \\ &= e^{(x+y)}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)\end{aligned}$$

- If such a diffeomorphism, $\Gamma$, exists, then

$$\Gamma_*\left(e^{(x+y)}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)\right) = \Gamma_*[X,Y] = [\Gamma_* X, \Gamma_* Y] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$$

However,

$$\Gamma_*\left(e^{(x+y)}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)\right)$$

cannot be the zero-vector field. In fact, it must be non-zero everywhere since $\Gamma_*$ is an isomorphism. Hence, no such $\Gamma$ exists.

- Let $\gamma(t) = (x(t), y(t))$ be an integral curve for $X+Y$. The associated system of ODE is

$$\frac{dx}{dt} = e^{y(t)}$$
$$\frac{dy}{dt} = e^{x(t)}$$

Note that the equation

$$e^{x(t)} = e^{y(t)}$$

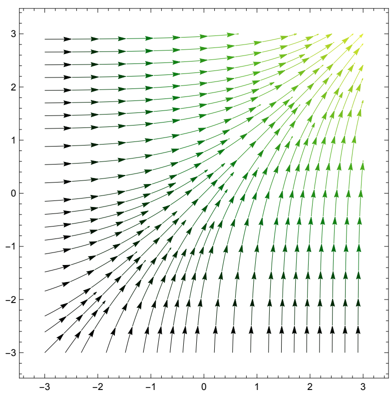
defines an integral curve, $\gamma$, that lies on the $y = x$ lines and is such that

$$\frac{dx}{dt} = e^{x(t)}$$

The solution to such an ODE is of the form

$$x(t) = -\log\left(\frac{1}{A-t}\right)$$

for some $A \in \mathbb{R}$. Clearly, $x(t)$ is not defines for all values of $t$. Hence, $X+Y$ is incomplete.




**Exercise 7.** The solution is given below:

- Recall that any orientation preserving isometry of $\mathbb{R}^2$ is of the form

$$g(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

for some $A \in \mathbf{SO}(2, \mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^2$. Therefore,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some $\theta \in [0, 2\pi)$. Write $\gamma(s) = (x(s), y(s))$. We have

$$(g\gamma(s))' = \begin{bmatrix} x'(s)\cos\theta - y'(s)\sin\theta \\ y'(s)\cos\theta + x'(s)\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix} = A\gamma'(s)$$

Define

$$\tilde{\mathsf{B}}(s) = (g\gamma(s))' \qquad \tilde{\mathsf{N}}(s) = \mathbb{J}\tilde{\mathsf{B}}(s)$$

We have

$$\begin{aligned}
\tilde{\mathsf{B}}'(s) &= \begin{bmatrix} x''(s)\cos\theta - y''(s)\sin\theta \\ y''(s)\cos\theta + x''(s)\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x''(s) \\ y''(s) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -k^{\pm}(s)y'(s) \\ k^{\pm}(s)x'(s) \end{bmatrix} \\
&= k^{\pm}(s) \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -y'(s) \\ x'(s) \end{bmatrix} \\
&= k^{\pm}(s)\tilde{\mathsf{N}}(s)
\end{aligned}$$

Hence, $g \circ \gamma$ and $\gamma$ have the same signed curvature.

- By construction, we have:

$$\begin{bmatrix} \mathsf{B}'(s) \\ \mathsf{N}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa^{\pm}(s) \\ -\kappa^{\pm}(s) & 0 \end{bmatrix} \begin{bmatrix} \mathsf{B}(s) \\ \mathsf{N}(s) \end{bmatrix}$$

This is the 2-D analog of the Frenet-Serre structure equations. An argument entirely analogous to that given in the next problem can now be used to solve the remaining parts. For a more elementary argument, see [Pre10, Theorem 2.26]

**Exercise 8.** The solution is given below:

- Note that

$$\|\gamma'(s)\| = 1 \iff \gamma'(s)\cdot\gamma'(s) = 1 \iff \gamma''(s)\cdot\gamma'(s) = 0$$

Hence $\gamma''(s)$ is orthogonal to $\gamma'(s)$ and hence $\mathsf{T}(s)$.

- We have

$$\|\mathsf{T}(s)\| = \|\mathsf{T}(s)\|\,\|\mathsf{N}(s)\|\sin\theta(s) = 1\cdot 1\cdot \sin(\pi/2) = 1$$

- It is clear that $(\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s))$ forms an orthonormal basis for $\mathbb{R}^3$ for each $s \in (a, b)$. Moreover, $(\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s))$ is a positively oriented basis since[9]

$$(\mathsf{T}(s)\times\mathsf{N}(s))\cdot\mathsf{B}(s) = \mathsf{B}(s)\cdot\mathsf{B}(s) = 1 > 0$$

[9]Here I use the result that a basis $(u, v, w)$ of $\mathbb{R}^3$ is positively oriented if $(u \times v)\cdot w > 0$, and negatively oriented if $(u \times v)\cdot w < 0$

- By definition, we have
$$\mathsf{T}'(s) = k(s)\mathsf{N}(s)$$
Since $\|\mathsf{N}(s)\| = 1$, we have $\mathsf{N}'(s) \cdot \mathsf{N}(s) = 0$. Hence,
$$\mathsf{N}'(s) = \alpha(s)\mathsf{T}(s) + \tau(s)\mathsf{B}(s)$$
Note that:
$$\begin{aligned}\mathsf{B}'(s) &= (\mathsf{T}(s) \times \mathsf{N}(s))' \\ &= \mathsf{T}'(s) \times \mathsf{N}(s) + \mathsf{T}(s) \times \mathsf{N}'(s) \\ &= \kappa(s)\mathsf{N}(s) \times \mathsf{N}(s) + \mathsf{T}(s) \times (\alpha(s)\mathsf{T}(s) + \tau(s)\mathsf{B}(s)) \\ &= \tau(s)\mathsf{T}(s) \times \mathsf{B}(s) \\ &= -\tau(s)\mathsf{N}(s)\end{aligned}$$
Similarly, we have
$$\begin{aligned}\mathsf{N}'(s) &= (\mathsf{B}(s) \times \mathsf{T}(s))' \\ &= \mathsf{B}'(s) \times \mathsf{T}(s) + \mathsf{B}(s) \times \mathsf{T}'(s) \\ &= -\tau(s)\mathsf{N}(s) \times \mathsf{T}(s) + \kappa(s)\mathsf{B}(s) \times \mathsf{N}(s) \\ &= \tau(s)\mathsf{T}(s) \times \mathsf{N}(s) - \kappa(s)\mathsf{N}(s) \times \mathsf{B}(s) \\ &= -\kappa(s)\mathsf{T}(s) + \tau(s)\mathsf{B}(s)\end{aligned}$$
Therefore, we have:
$$\begin{bmatrix}\mathsf{T}'(s) \\ \mathsf{N}'(s) \\ \mathsf{B}'(s)\end{bmatrix} = \begin{bmatrix}0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathsf{T}(s) \\ \mathsf{N}(s) \\ \mathsf{B}(s)\end{bmatrix}. \tag{5}$$
- Let $\mathsf{W}(s) = \mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s)$. Since $\|\mathsf{W}(s)\| = 1$, we have:
$$\mathsf{W}'(s) \cdot \mathsf{W}(s) = 0$$
Moreover, for $\mathsf{W}_1(s) \neq \mathsf{W}_2(s)$, we have:
$$\mathsf{W}_1(s) \cdot \mathsf{W}_2'(s) + \mathsf{W}_2(s) \cdot \mathsf{W}_1'(s) = (\mathsf{W}_1(s) \cdot \mathsf{W}_2(s))' = 0$$
The last equality follows since $\mathsf{W}_1(s) \cdot \mathsf{W}_2(s) = 0$ since $\mathsf{W}_1(s) \neq \mathsf{W}_2(s)$. Therefore,
$$\mathsf{W}_1(s) \cdot \mathsf{W}_2'(s) = -\mathsf{W}_2(s) \cdot \mathsf{W}_1'(s)$$
This shows that the the matrix must be skew-symmetric, which is indeed the case.
- Recall that any orientation preserving isometry of $\mathbb{R}^3$ is of the form
$$g(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

for some $A \in \mathbf{SO}(3, \mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^3$. Therefore, we have

$$\tilde{k}(s) = \|\tilde{\gamma}(s)''\| = \|(A\gamma(s) + \mathbf{b})''\| = \|A\gamma''(s)\| = \|\gamma''(s)\| = k(s)$$

The second last equality follows since $A$ preserves the Euclidean dot product and the Euclidean norm. Hence, the curvature is invariant under the action of an orientation-preserving isometry. Note that

$$\tilde{\mathsf{T}}(s) = A\mathsf{T}(s)$$

Since $k(s) = \tilde{k}(s)$, we also have that

$$\tilde{\mathsf{N}}(s) = A\mathsf{N}(s)$$

Since $A$ preserves angles, $A\mathsf{B}(s)$ is perpendicular to both $A\mathsf{T}(s) = \tilde{\mathsf{T}}(s)$ and $A\mathsf{N}(s) = \tilde{\mathsf{N}}(s)$. Hence $A\mathsf{B}(s) = \pm\tilde{\mathsf{B}}(s)$. Since $\gamma$ is orientation-preserving, we must have $A\mathsf{B}(s) = \tilde{\mathsf{B}}(s)$ Hence:

$$\tilde{\tau}(s) = \tilde{\mathsf{N}}'(s) \cdot \tilde{\mathsf{B}}(s) = (A\mathsf{N}'(s)) \cdot (A\mathsf{B}(s)) = \mathsf{N}'(s) \cdot \mathsf{B}(s) = \tau(s)$$

Hence, the torsion is invariant under the action of an orientation-preserving isometry.

- Let $A(s)$ denote the matrix appearing in the Frenet-Serret structure equations and consider the following vector:

$$\mathsf{E}(s) = \begin{bmatrix} \mathsf{T}(s) \\ \mathsf{N}(s) \\ \mathsf{B}(s) \end{bmatrix} \qquad \mathsf{E}'(s) = \begin{bmatrix} \mathsf{T}'(s) \\ \mathsf{N}'(s) \\ \mathsf{B}'(s) \end{bmatrix} \qquad \mathsf{E}(0) = \begin{bmatrix} \mathsf{T}(0) \\ \mathsf{N}(0) \\ \mathsf{B}(0) \end{bmatrix} := \mathbf{c}$$

WLOG, assume that $k(s)$ and $\tau(s)$ are defined on some symmetric interval about the origin, $(-a, a)$ for some $a > 0$. Consider the following initial value problem:

$$\mathsf{E}'(s) = A(s)\mathsf{E}(s) \qquad \mathsf{E}(0) = \mathbf{c} \tag{*}$$

Note that the initial value problem is a system of nine linear first order ODEs in the nine vectors that determine $\mathsf{E}(s)$. Since $k(s), \tau(s) \in \mathsf{C}^\infty(-a, a)$, we have $A(s)$ is a smooth matrix-valued function. By the existence and uniqueness of solutions for linear ODE, there exits an interval $0 \in I \subseteq (-a, a)$ and $\mathsf{E}(s) \in \mathbb{R}^9$ solving $(*)$. It can be checked that $E(s)$ defines an orthonormal frame for each $s \in I$. This will require solving another system of ODE and we will have to use the fact that the matrix appearing in the Fernet-Serre stucture equations is skew-symmetric. Details skipped.

Consider the curve $\mathbf{x}(t) : I \to \mathbb{R}^3$ defined by

$$\mathbf{x}(t) = \mathbf{x_0} + \int_{s \in I} \mathsf{T}(s) ds$$

where $\mathbf{x}(0) = \mathbf{x_0}$. We have

$$\mathbf{x}(s)'' = \mathsf{T}'(s) = k(s)\mathsf{N}(s)$$

It is also a simple matter to check that the torison of the curve $\mathbf{x}(s)$ is

$$\tau(s) = \mathsf{N}'(s) \cdot \mathsf{B}(s)$$

Therefore, $\mathbf{x}(s)$ is the unique curve such that $\mathbf{x}(0) = \mathbf{x_0}$ and has the required curvature and torsion for each $s \in I$.

More generally, assume that $\mathbf{x}_1(s)$ and $\mathbf{x}_2(s)$ are two space curves with the curvature and torsion functions $k(s)$ and $\tau(s)$ resp. Let the corresponding Frenet frames be $E_1(s)$ and $E_2(s)$ resp. Think fo $E_1(s)$ and $E_2(s)$ as 3-by-3 matrices with determinant one. Define:

$$A = E_1(0)E_2^{-1}(0) \qquad \mathsf{B} = \mathbf{x}_2(0) - A\mathbf{x}_1(0)$$

Consider the curve

$$\mathbf{x}_3(s) = A\mathbf{x}_1(s) + \mathsf{b}$$

Since $\det A = 1$, $\mathbf{x}_3$ is a curve with curvature $k(s)$ and $\tau(s)$. Since $\mathbf{x}_3(0) = \mathbf{x}_2(0)$, our discussion above implies that

$$A\mathbf{x}_1(s) + \mathsf{b} = \mathbf{x}_3(s) = \mathbf{x}_2(s)$$

- Consider the circular helix

$$\mathbf{x}(t) = (r \cos t, r \sin t, ht)$$

It can be checked that the circulat helix has constant curvature and torison

$$\kappa = \frac{r}{r^2 + h^2} \qquad \tau = \frac{h}{r^2 + h^2}$$

If $\tau = 0$, we have $h = 0$ and $r = \kappa$. If $\tau \neq 0$ We have:

$$r = \frac{\tau}{\kappa\tau^2 + \kappa} \qquad h = \frac{\tau^2}{\kappa\tau^2 + \kappa}$$

Hence for each $\tau$ and $\kappa > 0$, there exists a circular helix with curvature $\kappa$ and torsion $\tau$. By our above result, such a helix is defined uniquely up to an orientation-preserving isometry.

# MATH 740 HOMEWORK 3

W. GOLDMAN

(1) Let $M^m$ be a smooth manifold. Vector fields $X, Y \in \mathsf{Vec}(M) := \Gamma(\mathsf{T}M)$ *commute* if and only if $[X, Y] = 0$.
   (a) Suppose that $X_1, \dots, X_k \in \mathsf{Vec}(M)$ are vector fields whose values $X_1(x), \dots, X_k(x) \in T_x M$ are linearly independent. Local coordinates $(u^1, \dots, u^m)$ exist in a neighborhood of $x$ in which $X_i$ is the coordinate vector field:
   $$X_i = \frac{\partial}{\partial u^i}$$
   for $i = 1, \dots, k$. (Lee, Smooth manifolds, 231–236)
   (b) A *compact* manifold $M^m$ which admits $m$ *everywhere* linearly independent commuting vector fields is diffeomorphic to a torus. (Hint: Express $M$ as a quotient of $\mathbb{R}^m$ by a discrete subgroup $\Lambda < \mathbb{R}^m$.)
   (c) Furthermore suppose that $M$ admits a Riemannian structure $\mathsf{g} \in \Gamma(\mathsf{Sym}^2\mathsf{T}^*M)$ such that each $X_i$ is a *Killing vector field,* that is, an infinitesimal isometry (a vector field integrating to a one-parameter group $\mathbb{R} \longrightarrow \mathsf{Isom}(M, g)$ of isometries). Then $M$ is isometric to a *flat torus.*

(2) *Groups of isometries of Riemannian manifolds*
   (a) Let $(M, \mathsf{g})$ be a (connected) Riemannian manifold. Let $\phi \in \mathsf{Isom}(M, \mathsf{g})$ is an *isometry,* that is, a diffeomorphism
   $$M \xrightarrow{\phi} M$$
   such that $\phi^*\mathsf{g} = \mathsf{g}$. Suppose that $p \in M$ and $\phi(p) = p$. Then differential $(D\phi)_p$ of $\phi$ at $p$ acts on $\mathsf{T}_p(M)$:
   $$\mathsf{T}_p(M) \xrightarrow{(D\phi)_p} \mathsf{T}_{\phi(p)}(M) = \mathsf{T}_p(M)$$
   Suppose that $(D\phi)_p$ equals the identity map. Then $\phi$ is the identity. (Hint: It may be helpful if you first assume that $M$ is complete.)

*Date*: March 14, 2024.

(b) Deduce that if $M$ is compact, then $\mathsf{Isom}(M, \mathsf{g})$ is compact. (Hint: use the compactness of the orthogonal group $\mathsf{O}(n)$.)
(c) Find an example of a Riemannian manifold for which $\mathsf{Isom}(M, \mathsf{g})$ is *not* compact.

(3) Let $M^3$ be a smooth 3-manifold and let $\omega$ be a *contact form,* that is, a 1-form such that $\omega \wedge d\omega \neq 0$.
(a) Show that there exists a vector field $X$ such that $\omega(X) = 1$ and $\mathfrak{L}_X(\omega) = 0$.
(b) Show that every point $p \in M$ has an open neighborhood $U$ and *local coordinates* $(x, y, z)$ such that

$$\omega \;=\; x\,dy + dz.$$

(c) Let $M = \mathbb{R}^3$ and let $\omega$ be as above. Show that $\forall p, q \in M$, can be connected by a smooth path $\gamma(t)$ such that

$$\omega\big(\gamma'(t)\big) = 0.$$

(Hint: consider the projection

$$\mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \mapsto (x, y),$$

and the problem of lifting a smooth curve $\big(x(t), y(t)\big) \in \mathbb{R}^2$ to a curve $\gamma$ satisfying the above differential equation.)

(4) *Homogeneous Riemannian manifolds*
(a) Recall that a group $\Gamma$ acts *transitively* on a set $S :\Longleftrightarrow \forall p, q \in S, \exists \gamma \in \Gamma$ such that $p \stackrel{\gamma}{\mapsto} q$. A Riemannian manifold $(M, g)$ is *homogeneous* if $\mathsf{Isom}(M, g)$ acts transitively on $M$. Prove that a homogeneous Riemannian manifold is geodesically complete. (Hint: Show that there exists $\epsilon > 0$ such that all $p \in M$, the $\epsilon$-ball about $p$ is a normal neighborhood of $p$.)
(b) An *isometry* of a metric space $(M, d)$ is a map $M \xrightarrow{\phi} M$ which preserves distances:

$$d\big(\phi(p), \phi(q)\big) = d(p, q),$$

$\forall p, q \in M$. Find an example of an *incomplete* metric space $(M, d)$ for which its group of isometries acts transitively.[1]
(c) Find an example of a homogeneous Lorentzian manifold which is geodesically incomplete.

[1] I assigned this problem several years ago, I think, but I don't remember how to do it.

## 3. Homework 3

**Exercise 9.** The solution is given below:

- Nothing to prove.
-
-

**Exercise 10.** The solution is given below:

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-
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**Exercise 11.**

The solution is given below:

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-
-

**Exercise 12.** The solution is given below:

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-
-

# MATH 740 HOMEWORK 4

W. GOLDMAN

(1) Let $M^m$ be a smooth manifold.
   (a) Find two Riemannian structures which are not isometric but share the same Levi-Civita connection. (Hint: is every affine (that is, parallelism-preserving) transformation an isometry?)
   (b) *Prove or disprove:* The Levi-Civita connection of a (positive definite) Riemannian structure cannot be the Levi-Civita connection of a (strictly indefinite) Lorentzian structure.

(2) Suppose $\nabla$ is a torsionfree affine connection with curvature tensor $R$ defined by:

$$\mathsf{Vec}(M) \xrightarrow{R(X,Y):=\nabla_X\nabla_Y-\nabla_X\nabla_Y-\nabla_{[X,Y]}} \mathsf{Vec}(M)$$

for $X, Y \in \mathsf{Vec}(M)$.
   (a) Prove the *first Bianchi identity:* For vector fields $X, Y, Z \in \mathsf{Vec}(M)$,

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

   and express this identity in terms of the Cristoffel symbols $\Gamma^k_{ij}$.
   (b) Prove that if $\nabla$ is the Levi-Civita connection of a Riemannian structure $\mathsf{g}$, then

$$\mathsf{g}\big(R(X,Y)Z,W\big) = \mathsf{g}\big(R(Z,W)X,Y\big)$$

   for $X, Y, Z, W \in \mathsf{Vec}(M)$. In particular $R$ defines a self-adjoint tensorial mapping

$$\begin{aligned} \Lambda^2\mathsf{T}M &\longrightarrow \Lambda^2\mathsf{T}M \\ X\wedge Y &\longmapsto R(X,Y). \end{aligned}$$

*Date*: April 23, 2024.

(3) A Riemannian manifold $(M, \mathsf{g})$ is a *symmetric space* if, $\forall p \in M$, there is an isometry $\iota_p \in \mathsf{Isom}(M, \mathsf{g})$ such that:
   - $\iota_p(p) = p$.
   - The differential $(\mathsf{D}\iota_p)_p$ equals the inversion $-\mathbf{1}$ in the tangent space $\mathsf{T}_p M$.

   (a) Show that a symmetric space is geodesically complete.
   (b) Show that a symmetric space is *homogeneous:* its automorphism group acts transitively.
   (c) Show that Euclidean space, the sphere and hyperbolic space are all symmetric spaces.
   (d) Let $G$ be a compact Lie group. Then it admits a *bi-invariant* metric $\mathsf{g}$. Show that $(G, \mathsf{g})$ is a symmetric space where the inversion in the identity element $e \in G$ equals inversion

$$\begin{aligned} G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned}$$

   (e) Show that the geodesics in $(G, \mathsf{g})$ are the cosets of one-parameter subgroups.

## 4. Homework 4

**Exercise 13.** The solution is given below:

- Let $(M, g)$ be an Riemannian manifold. For $\lambda > 0$, $, \lambda g)$. These two Riemannian manifold are not isometric. by the formula

$$\begin{aligned}\Gamma^k_{ij;\lambda g} &= \frac{g^{kl}}{2\lambda}(\partial_i(\lambda g_{jl}) + \partial_j(\lambda g_{il}) - \partial_l(\lambda g_{ij})) \\ &= \frac{g^{kl}}{2\lambda}(\lambda \partial_i g_{jl} + \lambda \partial_j g_{il} - \lambda \partial_l g_{ij}) \\ &= \frac{g^{kl}}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\ &= \Gamma^k_{ij;g}\end{aligned}$$

  Here, $\Gamma^k_{ij;g}$ represents the Christoffel symbols for the metric $g$, and $\Gamma^k_{ij;\lambda g}(\lambda g)$. Hence, the Christoffel symbols symbols are the same. This implies that the Levi-Civita connection is also the same.
- This is false. Both $\mathbb{R}^4$ with the standard Euclidean metric, and $\mathbb{R}^4$ with the Minkowksi metric have the trivial Levi-Civita connection (all Christoffel symbols vanish).

**Exercise 14.** The solution is given below:

- We have

$$\begin{aligned}R(X, Y) &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \\ R(Y, Z) &= \nabla_Y \nabla_Z - \nabla_Z \nabla_Y - \nabla_{[Y,Z]}, \\ R(Z, X) &= \nabla_Z \nabla_X - \nabla_X \nabla_Z - \nabla_{[Z,X]}.\end{aligned}$$

  Then

$$\begin{aligned}R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X \\ + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y\end{aligned}$$

  Since our connection is torsion-free, we have that for vector fields $A$ and $B$

$$\nabla_A B - \nabla_B A = [A, B].$$

  Then rearranging terms and applying this fact, we get

$$\begin{aligned}R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) \\ &\quad - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\ &= \nabla_X [Y, Z] + \nabla_Y [Z, X] + \nabla_Z [X, Y] \\ &\quad - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y.\end{aligned}$$

Now we can use the fact that $\nabla$ is torsion-free again to rearrange and observe:

$$\begin{aligned}&(\nabla_X[Y,Z]-\nabla_{[Y,Z]}X)\\&+(\nabla_Y[Z,X]-\nabla_{[Z,X]}Y)\\&+(\nabla_Z[X,Y]-\nabla_{[X,Y]}Z)\\=&[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]\end{aligned}$$

By the Jacobi identity this is zero. Let $X=\partial_i, Y=\partial_j, Z=\partial_k$ We compute:

$$R(\partial_i,\partial_j)\partial_k=\nabla_{\partial_i}\nabla_{\partial_j}\partial_k-\nabla_{\partial_j}\nabla_{\partial_i}\partial_k-\nabla_{[\partial_i,\partial_j]}\partial_k=\nabla_{\partial_i}\nabla_{\partial_j}\partial_k-\nabla_{\partial_j}\nabla_{\partial_i}\partial_k$$

The last term vanishes since co-ordinate vector fields commute. Computing the first term we have

$$\begin{aligned}\nabla_{\partial_i}\nabla_{\partial_j}\partial_k&=\nabla_{\partial_i}\left(\Gamma^{\ell}_{jk}\partial_\ell\right)\\&=\partial_i(\Gamma^{\ell}_{jk})\partial_\ell+\Gamma^{\ell}_{jk}\nabla_{\partial_i}\partial_\ell\\&=\partial_i\Gamma^{\ell}_{jk}\partial_\ell+\Gamma^{\ell}_{jk}\Gamma^{s}_{i\ell}\partial_s\end{aligned}$$

Swap indices to get $\nabla_{\partial_j}\nabla_{\partial_i}\partial_k$:

$$\nabla_{\partial_j}\nabla_{\partial_i}\partial_k=\partial_j\Gamma^{\ell}_{ik}\partial_\ell+\Gamma^{\ell}_{ik}\Gamma^{s}_{j\ell}\partial_s$$

Therefore,

$$R(\partial_i,\partial_j)\partial_k=\underbrace{(\partial_i\Gamma^{\ell}_{jk}+\Gamma^{m}_{jk}\Gamma^{\ell}_{im}-\partial_j\Gamma^{\ell}_{ik}-\Gamma^{m}_{ik}\Gamma^{\ell}_{jm})}_{R^l_{ijk}}\partial_\ell$$

Similarly, we have:

$$R(\partial_j,\partial_k)\partial_i=\underbrace{(\partial_j\Gamma^{\ell}_{ki}+\Gamma^{m}_{ki}\Gamma^{\ell}_{jm}-\partial_k\Gamma^{\ell}_{ji}-\Gamma^{m}_{ji}\Gamma^{\ell}_{km})}_{R^l_{jki}}\partial_\ell$$

$$R(\partial_k,\partial_i)\partial_j=\underbrace{(\partial_k\Gamma^{\ell}_{ij}+\Gamma^{m}_{ij}\Gamma^{\ell}_{km}-\partial_i\Gamma^{\ell}_{kj}-\Gamma^{m}_{kj}\Gamma^{\ell}_{im})}_{R^l_{kij}}\partial_\ell$$

As a result

$$0=R(\partial_i,\partial_j)\partial_k+R(\partial_j,\partial_k)\partial_i+R(\partial_k,\partial_i)\partial_j=(R^l_{ijk}+R^l_{jki}+R^l_{kij})\partial_l$$

Hence,

$$R^l_{ijk}+R^l_{jki}+R^l_{kij}=0$$

for each $i,j,k,l$.

- Bianchi's first identity implies:

$$\begin{aligned}
&g(R(X,Y)Z,W)+g(R(Y,Z)X,W)+g(R(Z,X)Y,W)=0,\\
&g(R(Y,Z)W,X)+g(R(Z,W)Y,X)+g(R(W,Y)Z,X)=0,\\
&g(R(Z,W)X,Y)+g(R(W,X)Z,Y)+g(R(X,Z)W,Y)=0,\\
&g(R(W,X)Y,Z)+g(R(X,Y)W,Z)+g(R(Y,W)X,Z)=0.
\end{aligned}$$

  Then

$$\begin{aligned}
0 &= g(R(X,Y)Z,W)+g(R(Y,Z)X,W)+g(R(Z,X)Y,W)\\
&\quad+g(R(Y,Z)W,X)+g(R(Z,W)Y,X)+g(R(W,Y)Z,X)\\
&\quad+g(R(Z,W)X,Y)+g(R(W,X)Z,Y)\\
&\quad+g(R(X,Z)W,Y)+g(R(W,X)Y,Z)+g(R(X,Y)W,Z)+g(R(Y,W)X,Z)\\
&= g(R(X,Y)Z,W)+g(R(X,Y)W,Z)+g(R(Y,Z)X,W)+g(R(Y,Z)W,X)\\
&\quad+g(R(Z,W)Y,X)+g(R(Z,W)X,Y)+g(R(W,X)Z,Y)+g(R(W,X)Y,Z)\\
&\quad+g(R(Z,X)Y,W)+g(R(W,Y)Z,X)+g(R(X,Z)W,Y)+g(R(Y,W)X,Z).
\end{aligned}$$

  Using the result that

$$g(R(X,Y)Z,W)=-g(R(X,Y)W,Z), \tag{*}$$

  the first four pairs of terms cancel, leaving:

$$\begin{aligned}
0 &= g(R(Z,X)Y,W)+g(R(W,Y)Z,X)+g(R(X,Z)W,Y)+g(R(Y,W)X,Z)\\
&= -g(R(X,Z)Y,W)+g(R(W,Y)Z,X)+g(R(X,Z)W,Y)-g(R(W,Y)X,Z).
\end{aligned}$$

  Applying (*), we have

$$0=2g(R(X,Z)W,Y)-2g(R(W,Y)X,Z).$$

  In other words,

$$g(R(X,Z)W,Y)=g(R(W,Y)X,Z).$$

  In other words,

$$g(R(X,Y)Z,W)=g(R(Z,W)X,Y).$$

**Exercise 15.** The solution is given below:

- Sketch: Let $\gamma:(a,b)\to M$ be a geodesic. If $b<+\infty$, let $c$ be "sufficiently" near to $b$, and let $\iota_{\gamma(c)}$ be the symmetry at $\gamma(c)$. Since $\iota_{\gamma(c)}$ reverses geodesics through $\gamma(c)$, the domain of $\gamma$ can be extended beyond $b$.
- We already showed above that $M$ is geodesically complete. By Hopf-Rinow, any $p,q\in M$ can be joined by a unit-speed, length minimizing

geodesic, $\gamma|_{[0,d]} : [0, d] \to M$ is such that $\gamma(0) = p$, $\gamma(d) = q$ and $d = d_g(p, q)$. Then the symmetry $\iota_{\gamma(d/2)}$ at the point $\gamma(d/2)$ is an isometry and reverses geodesics, hence carries $q = \gamma(d)$ to $p = \gamma(0)$.

**Remark 4.1.** *For $p \in (M, g)$, let $\mathsf{Isop}_p(M, g)$ denote the set of isometries of $M$ that fix $p$. If $(M, g)$ is a homogenous Riemannian manifold, then it is easy to chech that*

$$\mathsf{Isop}_p(M, g) = F_{p,q}^{-1}\mathsf{Isop}_q(M, g)F_{p,q}$$

*for each $p, q \in (M, g)$ such that $F_{p,q}(p) = q$ and $F_{p,q}$ is a linear isometry. Moreover, it is clear that if an isometry in $\mathsf{Isop}_q(M, g)$ acts as inversion of $\mathsf{T}_qM$, then the corresponding isometry in $\mathsf{Isop}_p(M, g)$ also acts as inversion on $\mathsf{T}_pM$.*

- $\mathbb{R}^n$ is a symmetric space since for each $p \in \mathbb{R}^n$, the isometry

$$\iota_p(x) = 2p - x$$

is such that $\iota_p(p) = p$ and $\iota_p' = -\mathrm{Id}_{\mathbb{R}^n}$.

Since $\mathbb{S}^n$ is a homogeneous Riemannian manifold (with isometry group $\mathsf{O}(n + 1)$), it suffices to prove that $\mathbb{S}^n$ is symmetric at $N = (0, \cdots, 0, 1) \in \mathbb{S}^{n+1}$ based on the remark above. Consider the map

$$\iota_N(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_n, x_{n+1}).$$

defined on $\mathbb{R}^{n+1}$. It is a simple matter to check that $\iota_N$ restricts to a map from $\mathbb{S}^n$ to $\mathbb{S}^n$. Clearly, $\iota_N(N) = N$ and $(d\iota_N)_N = -\mathrm{Id}_{\mathsf{T}_X\mathbb{S}^n}$. The last equality follows since

$$\mathsf{T}_N\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$$

and

$$(d\iota_N)_N : \mathsf{T}_N\mathbb{R}^{n+1} \to \mathsf{T}_N\mathbb{R}^{n+1} \qquad (-x_1, \ldots, -x_n, 0).$$

Now simply note that $\iota_N$ restricts to the inversion map on $\mathsf{T}_N\mathbb{S}^n$. Hence, $(d\iota_N)_N$ is an isometry from $\mathsf{T}_N\mathbb{S}^n$ to $\mathsf{T}_N\mathbb{S}^n$ that satisfies the desired properties. Based on the remark above, $(d\iota_N)_p$ is indeed an isometry for each $p \in \mathbb{S}^n$.

We consider the hyperboloid model of hyperbolic space:

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n,1} : q(x, x) = -1 \text{ and } x^{n+1} > 0\}$$

Here $q(\cdot, \cdot)$ is the metric of signature $(n, 1)$. Since $\mathbb{H}^n$ is homogeneous (with isometry group $\mathsf{O}^+(n+1)$, the $(n+1)$-dimensional Lorentz group), it suffices to prove that $\mathbb{H}^n$ is symmetric at $P = (0, \cdots, 0, 1) \in \mathbb{H}^n$ based on the remark above. Consider the map

$$\iota_P(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_n, x_{n+1}).$$

defined on $\mathbb{R}^{n,1}$. It is a simple matter to check that $\iota_N$ restricts to a map from $\mathbb{H}^n$ to $\mathbb{H}^n$. Clearly, $\iota_P(P) = N$ and $(d\iota_N)_P = -\mathrm{Id}_{\mathsf{T}_X\mathbb{H}^n}$ based on arguments similar to given above for the sphere. Hence, $(d\iota_P)_P$ is an isometry from $\mathsf{T}_P\mathbb{H}^n$ to $\mathsf{T}_P\mathbb{H}^n$ that satisfies the desired properties. Based on the remark above, $(d\iota_N)_x$ is indeed an isometry for each $x \in \mathbb{H}^n$.

- Since a Lie group is a homogeneous Riemannian manifold, it suffices to prove that $G$ is symmetric at the identity, $e \in G$. Consider the inversion map
$$i : G \to G \qquad \iota(g) = g^{-1}$$
Clearly, $i(e) = e$. Moreover, we claim that the differential of $i$ at $e$
$$di_e : \mathsf{T}_eG \to \mathsf{T}_eG \qquad di_e(X) = -X$$
Consider the constant map
$$c : G \to G \qquad c(g) = e$$
$dc_e$ is clearly the zero map. $c$ can be thought of being given by the following composition:
$$g \mapsto (g, i(g)) \mapsto m(g, i(g)) = e$$
Therefore, we have[10]
$$0 = dd_e(X) = (X, di_e(X)) = X + di_e(X).$$
Therefore, $di_e(X) = -X$. Clearly, $di_e$ is an isometry since the inversion map is a linear isometry from $\mathsf{T}_eG$ to $\mathsf{T}_eG$.
- Skipped.

[10] We use the fact that $dm_{(e,e)}(X, Y) = X + Y$.

## References


[Lee12] John M. Lee. *Introduction to Smooth Manifolds*. Springer, 2012 (cit. on p. 32).

[Pre10] Andrew N. Pressley. *Elementary differential geometry*. Springer Science & Business Media, 2010 (cit. on p. 34).