SCHEME THEORY

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Abstract. These are notes on the theory of schemes covering sheaf theory, affine schemes, schemes, and the properties of schemes, with a view toward presenting the general theory of schemes. I wrote these notes at various stages during graduate school as part of an attempt to develop an understanding of modern algebraic geometry. There may be errors or typographical mistakes; corrections and suggestions are most welcome. Please send them to junaid.aftab1994@gmail.com.

Contents

1. Why Algebraic Geometry & Schemes?

Classical algebraic geometry is the study of affine algebraic sets, $X \subseteq \mathbb{C}^n$, given by the common zero set of a bunch of polynomials,

$$X = \{f_1(x) = \ldots = f_k(x) = 0\},$$

for some $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$.

Remark 1.1. *Classical algebraic geometry also studies projective algebraic sets.*

Classical algebraic geometry is captured by the slogan:

"algebra = geometry"

The slogan "algebra = geometry" is captured in the algebra-geometry correspondence. This correspondence forms a fundamental bridge between geometric objects and algebraic structures. This correspondence allows us to translate geometric problems into algebraic ones and vice versa. This duality is central to many powerful methods and results in affine algebraic geometry, enabling a deep interplay between geometry and algebra. When $\mathbb{K}$ is algebraically closed, this leads to the classical algebra-geometry correspondence:

$$\{\text{Affine algebraic subsets of } X \subseteq \mathbb{K}^n\} \longleftrightarrow \{\text{Radical Ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$

Scheme theory is the language of modern algebraic geometry. While the slogan "algebra = geometry" is already embodied in the classical algebra-geometry correspondence, why go further? One motivation lies in the following meta-principle:

A scheme is to a variety as an abstract
manifold is to an embedded submanifold of $\mathbb{R}^n$*.*

Recall the Whitney embedding theorem, which states that any smooth finite-dimensional manifold can be embedded in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Yet, smooth manifold theory is not about studying objects distinct from submanifolds of $\mathbb{R}^n$, but about understanding them in a way that emphasizes intrinsic properties, free from artifacts of any specific embedding. Similarly, scheme theory seeks to study classical affine and projective varieties intrinsically—beyond their realization as subsets of affine or projective space. This perspective not only clarifies foundational aspects but also provides the natural framework for advanced topics in algebraic geometry.

(1) **Intersection Theory**: Consider a basic example from intersection theory: the intersection of the line $y = 0$ and the parabola $y = x^2$. Classically, their intersection is the single point $(0, 0)$, but this misses an important feature—namely, tangency. From a scheme-theoretic perspective, the intersection is given by

$$\operatorname{Spec} \mathbb{R}[x, y]/(y, y - x^2) \cong \operatorname{Spec} \mathbb{R}[x]/(x^2),$$

which reflects the fact that the curves are tangent at the origin. Thus, the scheme-theoretic approach captures geometric information—like tangency—that the classical viewpoint overlooks.

(2) **Moduli Spaces**: Scheme theory also provides the natural language for constructing and understanding moduli spaces—parameter spaces that classify algebraic objects up to isomorphism. For instance, consider the problem of classifying algebraic curves of a fixed genus g. Classically, one might attempt to describe such families by explicitly writing down equations in projective space, but this quickly becomes unwieldy and fails to capture families of curves with desirable geometric structure (such as degenerations).

Remark 1.2. *In many important cases, moduli spaces are most naturally and accurately described using the language of stacks. This is particularly true when the objects being classified have non-trivial automorphisms. Stacks generalize schemes, allowing for a more flexible framework. In this sense, schemes serve as a foundational stepping stone toward understanding stacks, making scheme theory an essential prerequisite for moduli theory.*

Part 1. Spectrum of a Ring

The spectrum of a ring forms a fundamental bridge between commutative algebra and geometry. It provides the foundational framework through which algebraic data—such as rings and their ideals—are reinterpreted in geometric terms. In this part, we develop the concept of Spec R, the *spectrum* of a commutative ring R, and introduce the Zariski topology, which endows Spec R with a natural topological structure. The notion of the spectrum serves as the starting point for the theory of affine schemes, which are locally modeled on spectra of rings, and ultimately paves the way for the construction of general schemes.

2. Affine Algebraic Sets

The goal of affine algebraic geometry is to study the solution sets of polynomial equations in several variables over a fixed ground field. We introduce the main objects of study and outline the relationship between algebra and geometry.

Remark 2.1. *We denote affine n-space over a field $\mathbb{K}$ by*

$$\mathbb{A}^n_{\mathbb{K}} := \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{K} \text{ for } i = 1, \ldots, n\},$$

which is just $\mathbb{K}^n$ viewed geometrically. We will abbreviate $\mathbb{A}^n_{\mathbb{K}}$ as $\mathbb{A}^n$.

Let $\mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. We begin by defining an affine algebraic set.

Definition 2.2. For a subset $S \subseteq \mathbb{K}[x_1, \ldots, x_n]$ of polynomials, the affine zero locus of S,

$$\mathbb{V}(S) := \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\} \subseteq \mathbb{A}^n,$$

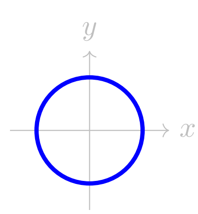
is an **affine algebraic set**.

Note that if $\mathfrak{a}$ is the ideal generated by S, then $\mathbb{V}(S) = \mathbb{V}(\mathfrak{a})$. Moreover, $\mathbb{V}(\mathfrak{a}) = \mathbb{V}(\sqrt{\mathfrak{a}})$ where $\sqrt{\mathfrak{a}}$ is the radical ideal of $\mathfrak{a}$.

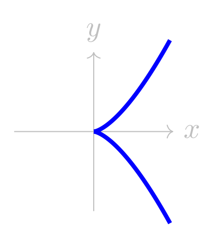
Remark 2.3. *If $S = \{f_1, \ldots, f_r\}$ is a finite set, we will write $\mathbb{V}(S) = \mathbb{V}(f_1, \ldots, f_r)$. Since $\mathbb{K}[x_1, \cdots, x_n]$ is a Noetherian ring, Hilbert's basis theorem (Proposition 27.6) implies that every $\mathbb{V}(S)$ is of the form $\mathbb{V}(f_1, \ldots, f_k)$ for some $f_1, \cdots, f_r \in \mathbb{K}[x_1, \cdots, x_n]$.*

Example 2.4. The following is a list of affine algebraic sets:

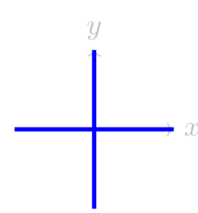
(1) Any point in $\mathbb{A}^n$ with $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is an affine algebraic set.
(2) Linear subspaces of $\mathbb{A}^n$ are algebraic sets.
(3) Let $\mathbb{K} = \mathbb{R}$. Some affine algebraic sets in $\mathbb{R}^2$ are shown below:



$\mathbb{V}(x^2 + y^2 - 1)$



$\mathbb{V}(y^2 - x^3 - 1)$



$\mathbb{V}(xy)$

Our goal is to study geometric properties of affine algebraic sets through their defining polynomials from an algebraic perspective. However, it is not sufficient to consider only the initially given polynomials, since they are not unique. For example,

$$\mathbb{V}(x^2 + y^2 - 1) = \mathbb{V}((x^2 + y^2 - 1)^2),$$

even though the defining expressions differ. This motivates the need to consider all polynomials that vanish on a given affine algebraic set—that is, its vanishing ideal.

Definition 2.5. Let $X \subseteq \mathbb{A}^n$ be any subset. The **ideal of** X is the set:

$$\mathbb{I}(X) := \{f \in \mathbb{K}[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in X\}.$$

Remark 2.6. $\mathbb{I}(X)$ *is indeed an ideal, as can be easily verified. In fact, it is a radical ideal.*

Example 2.7. Let $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ be a point. We claim that

$$\mathbb{I}(a) = (x_1 - a_1, \ldots, x_n - a_n).$$

(1) If $f \in \mathbb{I}(a)$, then $f(a) = 0$. This means that replacing each x_i by a_i in f gives zero, i.e., that f is zero modulo $(x_1 - a_1, \ldots, x_n - a_n)$. Hence $f \in (x_1 - a_1, \ldots, x_n - a_n)$.
(2) If $f \in (x_1 - a_1, \ldots, x_n - a_n)$, then $f = \sum_{i=1}^{n}(x_i - a_i)f_i$ for some $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$, and so certainly $f(a) = 0$, i.e., $f \in \mathbb{I}(a)$.

Note that we now have two distinct operations, $\mathbb{V}(\cdot)$ and $\mathbb{I}(\cdot)$. Moreover, these operations allow us to move and forth between subsets of $\mathbb{A}^n$ and subsets of $\mathbb{K}[x_1, \cdots, x_n]$.

$$\{\text{Subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{Subsets of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$X \mapsto \mathbb{I}(X)$$
$$\mathbb{V}(S) \leftarrow\!\shortmid S$$

Actually, $\mathbb{I}(X)$ is a radical ideal of of $\mathbb{K}[x_1, \cdots, x_n]$ and $\mathbb{I}(S)$ is an affine algebraic subset of $\mathbb{A}^n$. Hence, we have the following maps:

$$\{\text{Affine algebraic subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{Radical ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$X \mapsto \mathbb{I}(X)$$
$$\mathbb{I}(S) \leftarrow\!\shortmid S$$

This begs the question: are the operations $\mathbb{V}(\cdot)$ and $\mathbb{I}(\cdot)$ and inverses of each other? An investigation of this question is important since since it is in some sense the central question in affine algebraic geometry: is there a bijective correspondence between geometric objects (affine algebraic sets) and algebraic objects (radical ideals)?

Conjecture 2.8. *Let* $\mathbb{K}$ *be an algebraically closed field. Consider the operations* $\mathbb{V}(\cdot)$ *and* $\mathbb{I}(\cdot)$ *define above.*

$$\{\textit{Affine algebraic subsets of } \mathbb{A}^n\} \longleftrightarrow \{\textit{Radical ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$X \mapsto \mathbb{I}(X)$$
$$\mathbb{I}(S) \leftarrow\!\shortmid S$$

Then $\mathbb{V}(\cdot)$ *and* $\mathbb{I}(\cdot)$ *yield an inclusion-reversing bijective correspondence between affine algebraic sets of* $\mathbb{A}^n$ *and radical ideals* $\mathbb{K}[x_1, \cdots, x_n]$

We will prove Conjecture 2.8 in Proposition 3.6. For now, we assume the validity of the bijective correspondence between affine algebraic sets and their vanishing ideals, and explore some of its consequences. Throughout, we assume $\mathbb{K}$ is an algebraically closed field. If $X \subseteq \mathbb{A}^n$ is a fixed affine algebraic set, we are often interested in identifying polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ that take the same values at every point of X. This leads to the following definition:

Definition 2.9. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic subset. The **coordinate ring** of X is the quotient ring:

$$A(X) = \frac{\mathbb{K}[x_1, \cdots, x_n]}{\mathbb{I}(X)}$$

Remark 2.10. *In $A(X)$, we identify two polynomials $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ if and only if $f - g$ vanishes on X; that is, $f(x) = g(x)$ for all $x \in X$. Thus, an element $f \in A(X)$ can be viewed as a function $X \to \mathbb{K}$, given by evaluating a polynomial at points of X, where functions differing by a polynomial vanishing on X are considered equal.*

Given a fixed affine algebraic set X, one may focus on studying affine algebraic subsets of X. This motivates the following definition:

Definition 2.11. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic subset and $A(X)$ be the associated co-ordinate ring. For any $S \subseteq A(X)$, the **X-affine algebraic subsets** is the zero locus

$$\mathbb{V}_X(S) = \{x \in X : f(x) = 0 \text{ for all } f \in S\} \subseteq X$$

For subset $Y \subseteq X$,

$$\mathbb{I}_X(Y) = \{f \in A(X) : f(x) = 0 \text{ for all } x \in Y\} \trianglelefteq A(X),$$

is the ideal of all polynomials on X that vanish on Y.

The assumed bijective correspondence can now be refined as follows:

Conjecture 2.12. *Let $\mathbb{K}$ be an algebraically closed field. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic subset. There is an inclusion reversing bijective correspondence:*

$$\{\textit{Affine algebraic subsets of } X \subseteq \mathbb{A}^n\} \longleftrightarrow \{\textit{Radical ideals of } A(X)\}$$
$$Y \mapsto \mathbb{I}_X(Y)$$
$$\mathbb{V}_X(S) \leftarrow\!\shortmid S$$

The bijective correspondences described above imply that every algebraic operation on (radical) ideals admits a geometric interpretation.

We now illustrate this principle with several examples.

Example 2.13. We can give a geometric interpretation of various operations of ideals:

(1) Clearly, $\mathbb{V}_X(0) = X$ and $\mathbb{V}_X(A(X)) = \emptyset$.
(2) For any two ideals I, J be ideals in $A(X)$, note that $I + J$ is the ideal generated by $I \cup J$. We have:

$$\begin{aligned}
\mathbb{V}_X(I \cup J) &= \mathbb{V}_X(I + J) \\
&= \{x \in X : f(x) = 0 \text{ for all } f \in I \cup J\} \\
&= \{x \in X : f(x) = 0 \text{ for all } f \in I\} \cap \{x \in X : f(x) = 0 \text{ for all } f \in J\} \\
&= \mathbb{V}_X(I) \cap \mathbb{V}_X(J).
\end{aligned}$$

Algebraic Subsets	**Ideals of** $\mathbb{K}[x_1, \ldots, x_n]$
$\mathbb{A}^n$	(0)
$\emptyset$	$\mathbb{K}[x_1, \ldots, x_n]$
X	$A(X)$
$Y \cap Z$	$I + J$
$Y \cup Z$	$I \cap J$
$Y \subseteq Z$	$I : J$
$Y \cap Z = \emptyset$	$I + J = A(X)$

Algebra–geometry correspondence. Here $\mathbb{V}_X(I) = Y$ and $\mathbb{V}_X(J) = Z$.

Hence, the ideal $I \cup J$ corresponds to the union of the algebraic sets $\mathbb{V}_X(I)$ and $\mathbb{V}_X(J)$. In particular, if $I + J = A(X)$, then I and J are coprime ideals, and

$$\mathbb{V}_X(I) \cap \mathbb{V}_X(J) = \emptyset.$$

(3) For any two X-affine algebraic subsets $Y, Z \subseteq X$:

$$\begin{aligned}\mathbb{I}_X(Y \cup Z) &= \{f \in A(X) : f(x) = 0 \text{ for all } x \in Y \cup Z\} \\ &= \{f \in A(X) : f(x) = 0 \text{ for all } x \in Y\} \cap \{f \in A(X) : f(x) = 0 \text{ for all } x \in Z\} \\ &= \mathbb{I}_X(Y) \cap \mathbb{I}_X(Z).\end{aligned}$$

Hence, the ideal $\mathbb{I}_X(Y) \cap \mathbb{I}_X(Z)$ corresponds to the union of the affine algebraic subsets $Y, Z \subseteq X$.

(4) For any two X-affine algebraic subsets $Y, Z \subseteq X$:

$$\begin{aligned}\mathbb{I}_X(Y \setminus Z) &= \{f \in A(X) : f(x) = 0 \text{ for all } x \in Y \setminus Z\} \\ &= \{f \in A(X) : f(x)g(x) = 0 \text{ for all } x \in Y \text{ and } g \in \mathbb{I}_X(Z)\} \\ &= \{f \in A(X) : f \cdot \mathbb{I}_X(Z) \subseteq \mathbb{I}_X(Y)\} \\ &= \mathbb{I}_X(Y) : \mathbb{I}_X(Z)\end{aligned}$$

So taking the set-theoretic difference $Y \setminus Z$ corresponds to quotient ideals.

Given $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, we can also consider functions between affine algebraic subsets.

Definition 2.14. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. A **polynomial morphism** from X to Y is a set-theoretic map

$$f : X \to Y$$

such that there exist polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ satisfying

$$f(x) = (f_1(x), \ldots, f_m(x)) \in Y$$

for all $x \in X$.

Given the algebra-geometry correspondence discussed above, a natural question arises: what is the algebraic counterpart at the level of coordinate rings of a polynomial morphism between affine algebraic sets? The answer is provided by the following definition:

Definition 2.15. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine algebraic sets and $f : X \to Y$ be a polynomial morphism. Then f induces a ring homomorphism

$$\begin{aligned} \phi : A(Y) &\to A(X) \\ g &\mapsto g \circ f = g(f_1, \dots, f_m) \end{aligned}$$

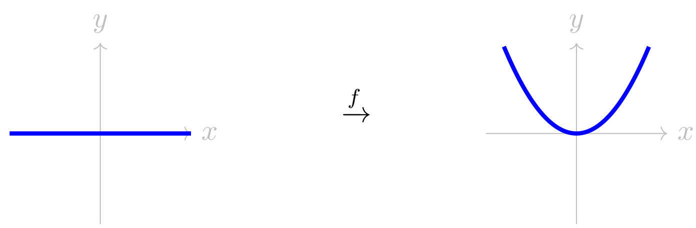
given by composing a polynomial function on Y with f to obtain a polynomial function on X.

Remark 2.16. *It is easy to check that the ϕ defined above is a $\mathbb{K}$-algebra homomorphism.*

Example 2.17. Let $\mathbb{K} = \mathbb{R}$, $X = \mathbb{A}^1$ (with coordinate x) and $Y = \mathbb{A}^2$ (with coordinates y_1 and y_2). Then $A(X) = \mathbb{R}[x]$ and $A(Y) = \mathbb{R}[y_1, y_2]$. Consider a polynomial morphism of affine algebraic sets,

$$\begin{aligned} f : X &\to Y \\ x &\mapsto (x, x^2) \end{aligned}$$

The image is obviously the standard parabola $Z = \mathbb{V}(y_2 - y_1^2)$.



The associated ring homomorphism $A(Y) = \mathbb{R}[y_1, y_2] \to \mathbb{R}[x] = A(X)$ is given by composing a polynomial function defined on Z with f, i.e., by plugging in x and x^2 for y_1 and y_2, respectively:

$$\begin{aligned} \mathbb{R}[y_1, y_2] &\to \mathbb{R}[x] \\ g &\mapsto g(x, x^2). \end{aligned}$$

Example 2.18. Let $f : X \to Y$ be a polynomial morphism of affine algebraic sets, and let $\phi : A(Y) \to A(X)$, $g \mapsto g \circ f$ be the associated map between the coordinate rings.

(1) For any X-affine algebraic subset Z, we have

$$\begin{aligned} \mathbb{I}(f(Z)) &= \{g \in A(Y) : g(f(x)) = 0 \text{ for all } x \in Z\} \\ &= \{g \in A(Y) : \phi(g) \in \mathbb{I}(Z)\} \\ &= \phi^{-1}(\mathbb{I}(Z)) \end{aligned}$$

Hence, taking images of X-affine algebraic subsets corresponds to the contraction of ideals.

(2) For any Y-affine algebraic subset Z, the zero locus of the extension $I(Z)$ by ϕ is

$$\begin{aligned} \mathbb{V}(\phi(\mathbb{I}(Z))) &= \{x \in X : g(f(x)) = 0 \text{ for all } g \in \mathbb{I}(Z)\} \\ &= f^{-1}(\{y \in Y : g(y) = 0 \text{ for all } g \in \mathbb{I}(Z)\}) \\ &= f^{-1}(\mathbb{V}(\mathbb{I}(Z))) \\ &= f^{-1}(Z) \end{aligned}$$

Hence, taking inverse images of Y-affine algebraic subsets corresponds to the extension of ideals.

Remark 2.19. *One can keep on asking similar questions:*

(1) *What X-affine algebraic sets correspond to maximal ideals in of $A(X)$?*
(2) *What X-affine algebraic sets correspond to prime ideals on of $A(X)$?*

We will answer this question in Proposition 3.20 by arguing that maximal ideals in $A(X)$ correspond to points in X and prime ideals in $A(X)$ correspond to irreducible[1] *X-affine algebraic sets.*

3. Algebra-Geometry Correspondence

We now set out to prove the algebra-geometry correspondence for affine algebraic sets. To do so, we first need to define a topology on the set of affine algebraic sets. This can be done via the following result:

Proposition 3.1. *The following properties are true for affine algebraic sets in $\mathbb{A}^n$:*

(1) *The empty set and the whole space are affine algebraic sets.*
(2) *The intersection of any family of affine algebraic sets is an affine algebraic set.*
(3) *The union of two affine algebraic sets is an affine algebraic set.*

Proof. The proof proceeds in the following steps:

(1) The empty set is $\emptyset = \mathbb{V}(1)$, and the whole space is $\mathbb{A}^n = \mathbb{V}(0)$.
(2) If $Y_\alpha = \mathbb{V}(T_\alpha)$ is any family of algebraic sets, then
$$\bigcap_\alpha Y_\alpha = \mathbb{V}\left(\bigcup_\alpha Y_\alpha\right),$$
so $\bigcap_\alpha Y_\alpha$ is also an affine algebraic set.
(3) Let $Y_1 = \mathbb{V}(T_1)$ and $Y_2 = \mathbb{V}(T_2)$, where $T_1, T_2 \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Then
$$Y_1 \cup Y_2 = \mathbb{V}(T_1T_2),$$
where T_1T_2 denotes the set of all finite sums of products fg with $f \in T_1$ and $g \in T_2$.
   (a) If $x \in Y_1 \cup Y_2$, then x is a zero of every polynomial in T_1T_2 since $x \in Y_1$ implies $f(x) = 0$ for all $f \in T_1$, and similarly for Y_2.
   (b) Conversely, suppose $x \in \mathbb{V}(T_1T_2)$ but $x \notin Y_1$. Then there exists $f \in T_1$ such that $f(x) \neq 0$. For any $g \in T_2$, since $(fg)(x) = f(x)g(x) = 0$ and $f(x) \neq 0$, it follows that $g(x) = 0$. Hence, $x \in Y_2$.

This completes the proof. □

Proposition 3.1 implies that the collection of affine algebraic sets is closed under arbitrary intersections and finite unions. This observation motivates the following definition of a topology:

Definition 3.2. The **(classical) Zariski topology** on $\mathbb{A}^n$ is defined by taking open subsets to be the complements of affine algebraic sets. This is a topology by Proposition 3.1.

[1] We will need to make sense of irreducible algebraic subsets as well.

What are the basis open sets in the (classical) Zariski topology? Let $U \subseteq \mathbb{A}^n$ be open in the Zariski topology. By definition, its complement U^c is an affine algebraic set, so

$$U^c = \mathbb{V}(I)$$

for some ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Since

$$U^c = \bigcap_{f \in I} \mathbb{V}(f),$$

it follows that

$$U = \bigcup_{f \in I} \mathbb{V}(f)^c := \bigcup_{f \in I} D(f),$$

where

$$D(f) = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}.$$

Hence, the collection

$$\{D(f) \mid f \in \mathbb{K}[x_1, \ldots, x_n]\}$$

forms a basis for the Zariski topology. Sets of this form are called *distinguished open sets.*

Example 3.3. Let $\mathbb{K}$ be an algebraically closed field. Consider the Zariski topology on $\mathbb{A}^1$. Since every ideal in $\mathbb{K}[x]$ is principal, every algebraic set is the zero locus of a single polynomial. Given that $\mathbb{K}$ is algebraically closed, every nonzero polynomial $f(x) \in \mathbb{K}[x]$ factors as

$$f(x) = c(x - a_1) \cdots (x - a_n),$$

for some $c, a_1, \ldots, a_n \in \mathbb{K}$. Thus,

$$\mathbb{V}(f) = \{a_1, \ldots, a_n\}.$$

Consequently, the (closed) algebraic sets in $\mathbb{A}^1$ are exactly the finite subsets (including the empty set) and the entire space (corresponding to the zero polynomial). In particular, this implies that the Zariski topology on $\mathbb{A}^1$ is not Hausdorff.

Example 3.4. Using properties of the classical Zariski topology, one can show that the zero loci of transcendental functions are not necessarily algebraic sets. Consider the set

$$X = \{(x, y) \in \mathbb{R}^2 \mid y - \cos x = 0\}$$

Assume that X is an affine algebraic set. Proposition 3.1 implies that $W = X \cap \{(x, 0) \mid x \in \mathbb{R}\}$ is an affine algebraic set since $\{(x, 0) \mid x \in \mathbb{R}\}$ is the zero set of $g(x, y) = y$. But W is an infinite subset of $\mathbb{R}$, and the only non-trivial affine algebraic subsets of $\mathbb{R}$ are finite subsets (Example 3.3). Its Zariski closure is

$$\bar{X} = \mathbb{V}(\mathbb{I}(X)) = \mathbb{V}(0) = \mathbb{R}^2$$

Proposition 3.5. *(*Hartshorne I.1.4) *If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.*

Proof. Consider the affine algebraic set $\mathbb{V}(y - x) \subseteq \mathbb{A}^2$. Clearly, $\mathbb{V}(y - x)$ is closed in the Zariski topology. However, $\mathbb{V}(y - x)$ is not closed in the product topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ equipped with the Zariski topology on each factor. Indeed, if $\mathbb{V}(y - x)$ were closed in the product topology, then its complement would be open. Any point in the complement would then be contained in a basis open set of the product topology. Since open sets in the Zariski topology on $\mathbb{A}^1$ are complements of finite sets, every basis open set in $\mathbb{A}^2$ is a product of cofinite sets and therefore must intersect $\mathbb{V}(y-x)$. This contradicts the assumption that the complement is open, proving that $\mathbb{V}(y-x)$ is not closed in the product topology. $\square$

We now use Hilbert's Nullstellensatz (Proposition 30.7), we are now able to prove the algebra-geometry correspondence (Conjecture 2.8).

Proposition 3.6. *Let $\mathbb{K}$ be an algebraically closed field. There is an inclusion-reversing bijection between radical ideals in $\mathbb{K}[x_1, \cdots, x_n]$and algebraic sets in $\mathbb{A}^n$. More specifically,*

1. If $T_1 \subseteq T_2$ are subsets of $\mathbb{K}[x_1, \cdots, x_n]$, then $\mathbb{V}(T_2) \subseteq \mathbb{V}(T_1)$.
2. If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, then $\mathbb{I}(Y_2) \subseteq \mathbb{I}(V_1)$.
3. For any ideal $\mathfrak{a}$ in $\mathbb{K}[x_1, \cdots, x_n]$, $\mathbb{I}(\mathbb{V}(a)) = \sqrt{\mathfrak{a}}$, the radical of a.
4. For any subset $Y \subseteq \mathbb{A}^n$, $\mathbb{V}(\mathbb{I}(Y)) = \overline{Y}$, the closure of Y.

Remark 3.7. *In what follows, let $R = \mathbb{K}[x_1, \cdots, x_n]$.*

Proof. (1), (2) are clear. The $\supseteq$ inclusion (3) is clear. For $\subseteq$ inclusion, assume that $f \notin \sqrt{\mathfrak{a}}$. We first argue that:

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{m} \ \mathfrak{m} \text{ maximal}} \mathfrak{m}$$

The $\subseteq$ inclusion is clear. For the opposite inclusion $\supseteq$, let $f \in R$ with $f \notin \sqrt{\mathfrak{a}}$; we have to find a maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$ with $f \notin \mathfrak{m}$. Consider the multiplicatively closed set $S = \{f^n : n \in \mathbb{N}\}$. Since $f \notin \sqrt{\mathfrak{a}}$, $\mathfrak{a} \cap S = \emptyset$. Hence, $\mathfrak{a}$ can be thought of as a prime ideal in the $S^{-1}R$. A standard Zorn's lemma argument then shows that there is a prime ideal $\mathfrak{p}$ with $\mathfrak{p} \supseteq \mathfrak{a}$ and $\mathfrak{p} \cap S = \emptyset$ such that $S^{-1}\mathfrak{p} := \mathfrak{p}_f$ is maximal. It only remains to show that $\mathfrak{p}$ is maximal in R. Consider the ring extension

$$k \to R/\mathfrak{p} \hookrightarrow (R/\mathfrak{p})_f = R_f/\mathfrak{p}_f$$

Note that the second map is, in fact, an inclusion since $R/\mathfrak{p}$ is an integral domain. Moreover, $R_f/\mathfrak{p}_f$ is a field since $\mathfrak{p}_f$ is maximal and finitely generated as a k-algebra. So $k \subseteq R_f/\mathfrak{p}_f$ is a finite field extension, and hence integral. But then $R/\mathfrak{p} \subset R_f/\mathfrak{p}_f$ is integral as well, which means that $R/\mathfrak{p}$ is a field since $R_f/\mathfrak{p}_f$ is. Hence, $\mathfrak{p}$ is maximal. Now there is then a maximal ideal $\mathfrak{m}$ with $\mathfrak{m} \supseteq \mathfrak{a}$ and $f \notin \mathfrak{m}$. By Hilbert's Nullstellensatz (Proposition 5.20), $\mathfrak{m}$ has to be of the form

$$\mathbb{I}(a) = (x_1 - a_1, \cdots, x_n - a_n)$$

for some point $a \in \mathbb{A}^n$. Now $\mathbb{I}(a) \supseteq \mathfrak{a}$ implies $\mathfrak{a} \in \mathbb{V}(I)$, and $f \notin I(\mathfrak{a})$ means $f(a) \neq 0$. Hence, $f \notin \mathbb{I}(\mathbb{V}(I))$. In particular, for any radical ideal $\mathfrak{a}$, we have $\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \mathfrak{a}$.

To prove (4), we note that $Y \subseteq \mathbb{V}(\mathbb{I}(Y))$, which is a closed set in the Zariski topology on $\mathbb{A}^n$, so clearly $\overline{Y} \subseteq \mathbb{V}(I(Y))$. On the other hand, let W be any closed set containing Y. Then $W = \mathbb{V}(\mathfrak{b})$ for some ideal $\mathfrak{b}$. So $\mathbb{V}(\mathfrak{b}) \supseteq Y$, and by (3), $\mathbb{I}(\mathbb{V}(\mathfrak{b})) \subseteq \mathbb{I}(Y)$. But certainly $\mathfrak{b} \subseteq \mathbb{I}(\mathbb{V}(\mathfrak{b}))$, so we have $W = \mathbb{V}(\mathfrak{b}) \supseteq \mathbb{V}(\mathbb{I}(Y))$. Thus, $\mathbb{V}(\mathbb{I}(Y)) = \overline{Y}$. In particular, if Y is an algebraic subset of $\mathbb{A}^n$, then $\mathbb{V}(\mathbb{I}(Y)) = Y$. $\square$

We immediately have the following corollary:

Corollary 3.8. *(Conjecture 2.8) Let $\mathbb{K}$ be an algebraically closed field. There is an inclusion-reversing bijective correspondence:*

$$\{\textit{Closed affine Algebraic Sets of } \mathbb{A}^n\} \longleftrightarrow \{\textit{Radical Ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$

$$X \longrightarrow \mathbb{I}(X)$$

$$\mathbb{I}(\mathfrak{a}) \longleftarrow \mathfrak{a}$$

We also have the following corollary:

Corollary 3.9. *(Conjecture 2.12) Let $\mathbb{K}$ be an algebraically closed field. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic subset. There is an inclusion reversing bijective correspondence:*

$$\{\textit{Closed affine algebraic subsets of } X \subseteq \mathbb{A}^n\} \longleftrightarrow \{\textit{Radical ideals of } A(X)\}$$
$$Y \mapsto \mathbb{I}_X(Y)$$
$$\mathbb{V}_X(S) \leftarrow\!\shortmid S$$

Remark 3.10. *Corollary 3.9 follows from Corollary 3.8. We omit details.*

Example 3.11. If we had not assumed $\mathbb{K}$ to be algebraically closed, then Corollary 3.8 would break down in the simple example with the the prime (and hence radical) ideal

$$\mathfrak{a} = (x^2 + 1) \subseteq \mathbb{R}[x]$$

The ideal has an empty zero locus in $\mathbb{A}^1$ (over $\mathbb{R}$ of course), so we would obtain $\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \mathbb{I}(\emptyset) = \mathbb{R}[x] \neq \sqrt{\mathfrak{a}} = \mathfrak{a}$.

Example 3.12. If X_1, X_2 are closed affine algebraic subsets, we can use Corollary 3.8 to prove that

$$\mathbb{I}(X_1 \cap X_2) = \sqrt{\mathbb{I}(X_1) + \mathbb{I}(X_2)}.$$

Indeed, we have

$$\mathbb{I}(X_1 \cap X_2) = \mathbb{I}(\mathbb{V}(\mathbb{I}(X_1)) \cap \mathbb{V}(\mathbb{I}(X_2))) = \mathbb{I}(\mathbb{V}(\mathbb{I}(X_1) + \mathbb{I}(X_2))) = \sqrt{\mathbb{I}(X_1) + \mathbb{I}(X_2)}.$$

Moreover, if $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$, we have

$$A(X) \cong A(X_1) \times A(X_2).$$

Indeed, we obtain in $A(X)$

$$\mathbb{I}(X_1) \cap \mathbb{I}(X_2) = \mathbb{I}(X_1 \cup X_2) = \mathbb{I}(X) = \{0\}.$$

On the other hand, from $X_1 \cap X_2 = \varnothing$, we have in $A(X)$

$$\sqrt{\mathbb{I}(X_1) + \mathbb{I}(X_2)} = \mathbb{I}(X_1 \cap X_2) = \mathbb{I}(\varnothing) = \langle 1 \rangle,$$

and thus also $\mathbb{I}(X_1) + \mathbb{I}(X_2) = \langle 1 \rangle$. By the Chinese Remainder Theorem, we conclude that

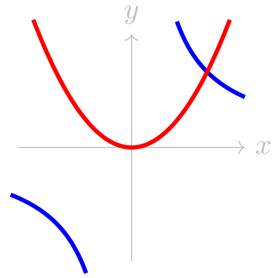
$$A(X) \cong A(X)/\mathbb{I}(X_1) \times A(X)/\mathbb{I}(X_2) \cong A(X_1) \times A(X_2).$$

Remark 3.13. *In light of Corollary 3.8 and Example 3.11, we almost exclusively assume that $\mathbb{K}$ is an algebraically closed field. In particular, we will work over $\mathbb{K} = \mathbb{C}$.*

We have now established a correspondence between affine algebraic sets of $\mathbb{A}^n$ and radical ideals of $\mathbb{K}[x_1, \cdots, x_n]$. This correspondence allows us us translate from thinking about affine algebraic sets as from either a geometric or an algebraic vantage point. Before moving on, let's look at an example:

Example 3.14. (*Hartshorne I.1.1*) Let Y_1 be the plane curve $y = x^2$ and let Y_2 be the plane curve $xy = 1$. The real locus of the plane curves is drawn below[2]. We claim that

[2] We shall only be able to plot the real locus of affine algebraic sets.



$A(Y_1) = \mathbb{C}[x,y]/(y-x^2)$ is isomorphic to polynomial ring in one variable over $\mathbb{C}$. Consider the following morphism:

$$\phi : \mathbb{C}[x,y] \longrightarrow \mathbb{C}[t]$$
$$x \mapsto t$$
$$y \mapsto t^2$$

The map is clearly a $\mathbb{C}$-algebra morphism. Note that the polynomial $y - x^2$ is contained in the kernel of ϕ. Therefore, ϕ descends to the map:

$$\overline{\phi} : \mathbb{C}[x,y]/(y-x^2) \longrightarrow \mathbb{C}[t]$$

It is easily seen that the map:

$$\phi : \mathbb{C}[t] \longrightarrow \mathbb{C}[x,y]$$
$$t \mapsto \bar{x}$$

is a $\mathbb{C}$-algebra morphism that is the inverse of ϕ. Therefore,

$$\mathbb{C}[x,y]/(y-x^2) \cong \mathbb{C}[t].$$

On the other hand, $A(Y_2) = \mathbb{C}[x,y]/(y-1/x)$ is not isomorphic to polynomial ring in one variable over $\mathbb{C}$. In the coordinate ring, x has a unit, namely $y = 1/x$. However, the indeterminate of a polynomial ring in one variable over $\mathbb{C}$ is never a unit.

Using the algebra-geometry correspondence, we can characterize irreducible affine algebraic sets.

Definition 3.15. Let X be a topological space. A non-empty subset $Y \subseteq X$ is **irreducible** if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper closed subsets of Y.

Example 3.16. Let $\mathbb{K}$ be an algebraically closed field. The following is a basic list of examples of irreducible and reducible affine algebraic sets:

(1) If $\mathbb{K}$ is infinite, $\mathbb{A}^1$ is irreducible in the Zariski topology, because its only proper closed subsets are finite, yet $\mathbb{A}^1$ is infinite[3].

(2) The affine algebraic $\mathbb{V}(xyz) \subseteq \mathbb{A}^3$ is not irreducible, as it can be written as a union of three coordinate planes:

$$\mathbb{V}(xyz) = \mathbb{V}(x) \cup \mathbb{V}(y) \cup \mathbb{V}(z).$$

Proposition 3.17. (Hartshorne I.1.6) *Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then $\overline{Y}$ is also irreducible.*

[3]Having $\mathbb{K}$ be infinite is crucial here.

Proof. The proof has been commented out as it follows from standard results in point-set topology. □

With the notion of irreducible sets in a topological space in mind, we can now define the concept of an affine variety which are irreducible algebraic sets.

Definition 3.18. Let $\mathbb{K}$ be an algebraically closed field. An **affine variety** is an irreducible affine algebraic set of $\mathbb{A}^n$

Remark 3.19. *An affine variety cannot be written as a union of two non-empty affine algebraic sets.*

Proposition 3.20. *Let $\mathbb{K}$ be an algebraically closed field, and let $Y \subseteq \mathbb{A}^n$ be an affine algebraic set. Y is as affine variety if and only if $\mathbb{I}(Y)$ is a prime ideal.*

Proof. Assume Y is irreducible. If $fg \in \mathbb{I}(Y)$, then using that $\mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$, we have

$$Y = Y \cap \mathbb{V}(fg) = (Y \cap \mathbb{V}(f)) \cup (Y \cap \mathbb{V}(g)),$$

both being closed subsets of Y. Since Y is irreducible, we have either $Y = Y \cap \mathbb{V}(f)$, in which case $Y \subseteq \mathbb{V}(f)$, or $Y \subseteq \mathbb{V}(g)$. Hence either $f \in \mathbb{I}(Y)$ or $g \in \mathbb{I}(Y)$. Conversely, assume that $\mathbb{I}(Y)$ is a prime ideal and $Y = Y_1 \cup Y_2$. If $Y = Y_1$, we are done. Hence, assume that $Y \neq Y_1 := \mathbb{V}(\mathfrak{a}_1)$. Then there is a $f_1 \in \mathfrak{a}_1$ and $y \in Y$ such that $f_1(y) \neq 0$. Writing $Y_2 := \mathbb{V}(\mathfrak{a}_2)$, we have that for every $f_2 \in \mathfrak{a}_2$, $f_1 f_2$ vanishes on Y and hence $f_1 f_2 \in \mathbb{I}(Y)$. Since $f_1 \notin \mathbb{I}(Y)$ and $\mathbb{I}(Y)$ is a prime ideal, we have that $f_2 \in \mathbb{I}(Y)$. This implies that $\mathfrak{a}_2 \subseteq \mathbb{I}(Y)$. But this implies that

$$Y \subseteq \mathbb{V}(\mathbb{I}(Y)) \subseteq \mathbb{V}(\mathfrak{a}_2) = Y_2.$$

Hence, $Y = Y_2$ and we are done. □

We have shown the following bijection:

$\{$Closed, irreducible affine Algebraic Sets of $\mathbb{A}^n\} \longleftrightarrow \{$Prime Ideals of $\mathbb{K}[x_1, \cdots, x_n]\}$

If $X \subseteq \mathbb{A}^n$ is an X-affine algebra set, we also have a bijection:

$\{$Closed, irreducible affine Algebraic Subsets of $X \subseteq \mathbb{A}^n\} \longleftrightarrow \{$Prime Ideals of $A(X)\}$

Example 3.21. Let $\mathbb{K}$ be an algebraically closed field. The following is a basic list of affine varieties:

(1) $\mathbb{A}^n$ is irreducible since it corresponds to (0) which is a prime ideal (0) as $\mathbb{K}$ is a field.
(2) Let $\mathbb{K} = \mathbb{C}$ and $X = \mathbb{V}(y^2 - x^3 - 1)$. We claim that X is irreducible by showing that the ideal $\mathbb{I}(y^2 - x^3 - 1)$ is prime in $\mathbb{C}[x, y]$. We consider the quotient ring:

$$R = \mathbb{C}[x, y]/(y^2 - x^3 - 1) \cong \mathbb{C}[x][y]/(y^2 - x^3 - 1)$$

Suppose $y^2 - x^3 - 1$ is reducible in $\mathbb{C}[x][y]$. Then it must factor as:

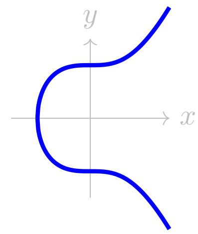
$$y^2 - x^3 - 1 = (y - f(x))(y - g(x))$$

for some $f(x), g(x) \in \mathbb{C}[x]$. This is because $y^2 = x^3 + 1$ in R. Expanding:

$$(y - f(x))(y - g(x)) = y^2 - (f(x) + g(x))y + f(x)g(x)$$

Comparing coefficients with $y^2 - x^3 - 1$, we get:

$$f(x) + g(x) = 0$$
$$f(x)g(x) = -x^3 - 1$$



$\mathbb{V}(y^2 - x^3 - 1)$

From the first equation, $g(x) = -f(x)$. Substituting into the second:

$$f(x)(-f(x)) = -x^3 - 1 \Longrightarrow -f(x)^2 = -x^3 - 1 \Longrightarrow f(x)^2 = x^3 + 1$$

This implies that $x^3 + 1$ must be a perfect square in $\mathbb{C}[x]$. However:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

which is not a square in $\mathbb{C}[x]$. Hence, $\mathbb{I}(y^2 - x^3 - 1)$ is a prime ideal in $\mathbb{C}[x, y]$.

Remark 3.22. *We have an additional bijection correspondences:*

$$\{\textit{Points in } \mathbb{A}^n\} \longleftrightarrow \{\textit{Maximal Ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$\{\textit{Points of } X \subseteq \mathbb{A}^n\} \longleftrightarrow \{\textit{Maximal Ideals of } A(X)\}$$

Here $X \subseteq \mathbb{A}^n$ is an affine algebra set. This can be easily proven given what we know now.

4. Morphisms of Affine Algebraic Sets

We first study the case of morphisms from an affine algebraic set to the affine algebraic set $\mathbb{A}^1 = \mathbb{K}$. Classical affine algebraic geometry studies the zero loci of polynomial functions, which correspond to morphisms from $\mathbb{A}^n$ to $\mathbb{A}^1$, or more generally, from an affine algebraic set X to $\mathbb{A}^1$, that is, elements of the coordinate ring $A(X)$. However, a broader class of functions is also important in this setting: rational functions. These only make sense in a restricted context, since for an affine algebraic set X, the coordinate ring $A(X)$ is not, in general, an integral domain. As shown in Proposition 3.20, this is the case if and only if X is an affine variety. Hence, we can make the following definition.

Definition 4.1. Let X be an affine variety, and let $A(X)$ denote its coordinate ring. The function field of X, denoted $K(X)$, is defined to be the field of fractions of $A(X)$:

$$K(X) := \text{Frac}(A(X)).$$

Elements of $K(X)$ are called rational functions on X. A rational function $f \in K(X)$ is said to be regular at a point $x \in X$ if there exists $g, h \in A(X)$ such that $f = g/h$ and $h(x) \neq 0$.

Remark 4.2. *One can check that Definition 4.1 is independent of the choice of representative fraction.*

For $x \in X$, the set of rational functions that are regular at x forms a subring of $K(X)$, called the local ring of X at x:

$$\mathcal{O}_{X,x} := \{f \in K(X) \mid f \text{ is regular at } x\}$$

The ring $\mathcal{O}_{X,x}$ is indeed a local ring, with maximal ideal

$$\mathfrak{m}_x := \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}.$$

In fact, if $x = (a_1, \dots, a_n) \in X$, and we write $\mathfrak{m}_x = (x_1 - a_1, \dots, x_n - a_n)$ for the maximal ideal in $A(X)$, then

$$\mathcal{O}_{X,x} = A(X)_{\mathfrak{m}_x}.$$

To see this, note that since $A(X)$ is an integral domain, the localization $A(X)_{\mathfrak{m}_x}$ is naturally a subring of $K(X)$. An element $f \in K(X)$ lies in $A(X)_{\mathfrak{m}_x}$ if and only if it can be written as a/b with $b \notin \mathfrak{m}_x$, which is equivalent to f being regular at x. If $U \subseteq X$ is an open set, we define

$$\mathcal{O}_X(U) := \{f \in K(X) \mid f \text{ is regular at every point in } U\} = \bigcap_{x \in U} \mathcal{O}_{X,x}.$$

Note that each $\mathcal{O}_X(U)$ is a sub-$\mathbb{K}$-algebra of $K(X)$. Moreover, if $V \subseteq U$, then any $f \in K(X)$ that is regular on U is also regular on V, so there is an inclusion

$$\mathcal{O}_X(U) \subseteq \mathcal{O}_X(V).$$

Remark 4.3. *We will later see that $\mathcal{O}_X$ is a sheaf.*

Proposition 4.4. *Let X be an affine variety.*

(1) *If a rational function $g \in K(X)$ is regular at every point of X, then g is a polynomial function. In other words,*

$$\mathcal{O}_X(X) = A(X).$$

(2) *If $f \in A(X)$ and $D(f) := \{x \in X \mid f(x) \neq 0\}$, then*

$$\mathcal{O}_X(D(f)) = A(X)_f.$$

Proof. (1) follows from (2) by taking $f = 1$. Clearly $A(X)_f \subseteq \mathcal{O}_X(D(f))$. Conversely, given $g \in K(X)$, define the ideal

$$I_g := \{b \in A(X) \mid bg \in A(X)\}.$$

This ideal has the property that g is regular at a point $x \in X$ if and only if $x \notin \mathbb{V}(I_g)$. Note that $x \notin \mathbb{V}(I_g)$ if and only if there exists some $b \in I_g$ with $b(x) \neq 0$, which is equivalent to g being of the form $g = a/b$ with $b(x) \neq 0$. Therefore, if g is regular on all of $D(f)$, it follows that $\mathbb{V}(I_g) \subseteq \mathbb{V}(f)$. By Proposition 5.20, we conclude that $f^n \in I_g$ for some $n > 0$. But then $f^n g \in A(X)$, which shows that $g \in A(X)_f$. □

Remark 4.5. *Proposition 4.4 is a sort of 'local-to-global principle': being regular is a local condition, which has to be verified near every point, but the conclusion is that a rational function which is regular at every point can be represented globally by a polynomial function.*

Example 4.6. Let $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{V}(x)$. Then $K(\mathbb{A}^1) = \mathbb{K}(x)$ and $\mathcal{O}_{\mathbb{A}^1}(U) = \mathbb{K}[x, x^{-1}]$.

We now turn to the notion of morphisms between affine varieties. We define morphisms between affine varieties in a way that captures the algebraic structure encoded by regular functions. To do this, we adopt a functorial perspective: a morphism of varieties will be defined in terms of how it pulls back regular functions. The utility of this functorial definition will become apparent later on when we derive an appropriate equivalence of categories.

Definition 4.7. Let X, Y be affine varieties. A morphism between X and Y is a continuous map $f \colon X \to Y$ such that the pullback of any regular function is again regular. That is, for every open set $V \subseteq Y$ and every $g \in \mathcal{O}_Y(V)$, the composition $g \circ f$ lies in $\mathcal{O}_X(f^{-1}(V))$.

Remark 4.8. *We write the pullback of f as f^*.*

Given n regular functions $f_1, \dots, f_n \in A(X)$, we can define a morphism

$$\begin{aligned} f\colon X &\to \mathbb{A}^n \\ x &\mapsto (f_1(x), \dots, f_n(x)). \end{aligned}$$

Let's check that f is continuous. Let $W = \mathbb{V}(g_1, \dots, g_r) \subseteq \mathbb{A}^n$ be a closed set., then

$$f^{-1}(W) = \{x \in X \mid g_i(f_1(x), \dots, f_n(x)) = 0 \text{ for all } i = 1, \dots, r\} = \mathbb{V}(f^*(g_1), \dots, f^*(g_r)),$$

which is closed in X. Let $g \in \mathbb{K}(y_1, \dots, y_n)$ be a rational function on $\mathbb{A}^n$, and assume g is regular on an open set $V \subseteq \mathbb{A}^n$. Let $x \in f^{-1}(V)$ and set $y = f(x)$. Locally around y, we may write $g = a/b$ where $a, b \in \mathbb{K}[y_1, \dots, y_n]$. Then in a neighborhood of $x \in f^{-1}(V)$, we have

$$f^*(g)(x) = \frac{a(f_1(x), \dots, f_n(x))}{b(f_1(x), \dots, f_n(x))}.$$

After expanding, this can be written as a quotient of polynomials where the denominator does not vanish at x. Hence $f^*(g) \in \mathcal{O}_X(f^{-1}(V))$, and f is a morphism. In fact, we can now argue that all morphisms with target affine variety $\mathbb{A}^n$ are of this form:

Proposition 4.9. *Let X be an affine variety. Then every morphism $f\colon X \to \mathbb{A}^n$ is of the form*

$$\begin{aligned} f\colon X &\to \mathbb{A}^n \\ x &\mapsto (f_1(x), \dots, f_n(x)), \end{aligned}$$

where $f_i \in A(X)$ for $i = 1, \dots, n$.

Proof. Let $f\colon X \to \mathbb{A}^n$ be a morphism. The coordinate functions $y_1, \dots, y_n$ are regular functions on $\mathbb{A}^n$, so their pullbacks $f^*(y_i)$ are regular on X. The morphism defined by these functions,

$$\begin{aligned} X &\to \mathbb{A}^n \\ x &\mapsto \big(f^*(y_1)(x), \dots, f^*(y_n)(x)\big). \end{aligned}$$

coincides with f, since both are determined by the same set of regular functions. □

Grothendieck emphasized studying affine algebraic sets via their coordinate rings rather than the sets themselves. This is also the philosophy underlying the algebra-geometry correspondence: the geometry of X reflects the algebra of its coordinate ring

$$A(X) = \mathbb{K}[x_1, \dots, x_n]/\mathbb{I}(X),$$

and vice versa. This can be formalized by arguing that we have an appropriate equivalence of categories between affine algebraic sets and finitely generated reduced $\mathbb{K}$-algebras. These categories are defined as follows:

1. The category AffVar of affine varieties and morphisms of affine algebraic varieties.
2. The category $\mathsf{fgAlg}_{\mathbb{K}}^{\mathsf{Dom}}$ of finitely-generated $\mathbb{K}$-algebras that are integral domains and morphisms of $\mathbb{K}$-algebras.

Proposition 4.10. *Let $\mathbb{K}$ be an algebraically closed field. The categories AffVar and $\mathsf{fgAlg}_{\mathbb{K}}^{\mathsf{Dom}}$ are equivalent.*

PROOF. Consider the functor that assigns to each affine variety $X \subseteq \mathbb{A}^n$ its coordinate ring $A(X) = \mathbb{K}[x_1, \ldots, x_n]/\mathbb{I}(X)$. Clearly, $A(X)$ is finitely generated as a $\mathbb{K}$-algebra since $\mathbb{K}[x_1, \ldots, x_n]$ is finitely generated. Moreover, $A(X)$ is reduced because $\mathbb{I}(X)$ is a radical ideal. The functor is essentially surjective. Indeed, if $A \in \mathsf{fgAlg}_{\mathbb{K}}^{\mathsf{Dom}}$ we can choose a presentation

$$A = \mathbb{K}[x_1, \ldots, x_n]/\mathfrak{p}$$

for some ideal $\mathfrak{p}$. Since A is an integral domain, $\mathfrak{p}$ is a prime ideal. Then $X = \mathbb{V}(\mathfrak{p}) \subseteq \mathbb{A}^n$ is an affine variety with $A(X) = A$. We claim that

$$\begin{aligned} \mathrm{Hom}_{\mathsf{AffVar}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathsf{fgAlg}_{\mathbb{K}}^{\mathsf{Dom}}}(A(Y), A(X)) \\ f &\mapsto f^* \end{aligned}$$

is a bijection. We first show that the map is surjective. Let $\phi\colon A(Y) \to A(X)$ be a morphism of $\mathbb{K}$-algebras. Suppose $Y \subseteq \mathbb{A}^n$, and let $y_1, \ldots, y_n$ be the coordinate functions on $\mathbb{A}^n$. Define $f_i := \phi(y_i)$ for $i = 1, \ldots, n$. Then the functions $f_1, \ldots, f_n \in A(X)$ define a morphism

$$\begin{aligned} f\colon X &\longrightarrow \mathbb{A}^n \\ x &\mapsto \big(f_1(x), \ldots, f_n(x)\big) \end{aligned}$$

This induces a map $\mathbb{K}[x_1, \cdots, x_n] \to A(X)$ via the pullback (Definition 2.15). More explicitly for $h \in \mathbb{K}[x_1, \cdots, x_n]$, we have

$$f^*(h)(x) = h(f(x)) = h(f_1(x), \ldots, f_n(x)) = \phi\big(h(y_1, \ldots, y_n)\big)(x)$$

The last equality holds because both sides agree on the generators $y_1, \ldots, y_n$, taking values $f_1, \ldots, f_n$, respectively. This shows that

$$f^*(h) = \phi(h) = 0 \quad \text{for every } h \in \mathbb{I}(Y),$$

since h is zero in $A(Y)$. Therefore, the image of f is contained in $Y = \mathbb{V}(\mathbb{I}(Y))$. This shows that f^* factors through $\mathbb{A}(Y) = K[x_1, \cdots, x_n]/\mathbb{I}(Y)$. Hence, the map is surjective. The map is also injective since $f^* = \phi$. □

5. Spectrum of a Ring

Recall the classical algebra-geometry correspondence (Proposition 3.6). There are some limitations of this correspondence:

(1) The bijection isn't natural. since morphisms of affine (projective) algebraic sets assumes an embedding of affine (projective) algebraic sets in some underlying affine (projective) space.
(2) The classical algebra-geometry correspondence for affine (projective) algebraic sets holds for algebraically closed fields. How does one study the analog of affine (projective) algebraic sets over non-algebraically closed fields?

We address these limitations by providing an intrinsic characterization of relevant concepts for affine algebraic sets. The classical algebra-geometry correspondence states that there we have bijections:

$$\begin{aligned} \{\text{Points of } \mathbb{A}^n\} &\longleftrightarrow \{\text{Maximal ideals of } \mathbb{K}[x_1, \cdots, x_n]\} \\ \{\text{Closed irreducible affine algebraic sets } \mathbb{A}^n\} &\longleftrightarrow \{\text{Prime ideals of } \mathbb{K}[x_1, \cdots, x_n]\} \end{aligned}$$

We can now attempt to generalize to the case of an arbitrary commutative ring, R. Previously, we have kept track of the data consisting of all irreducible algebraic subsets of

$\mathbb{A}^n$, which correspond to prime ideals of $R = \mathbb{K}[x_1, \ldots, x_n]$. This leads us to the following definition:

Definition 5.1. Let R be a commutative ring. The **spectrum of a ring** R, denoted as $\operatorname{Spec} R$, is the set of all prime ideals of R.

Remark 5.2. *All rings, unless otherwise specified, will be commutative with an identity. From now on, we shall use the phrase 'let R be a ring.' We will use the phrases the spectrum of a ring and affine schemes interchangeably to refer to the set* $\operatorname{Spec} R$. *We will justify the use of the phrase affine schemes in Section 14 when we formally define schemes.*

Remark 5.3. *The construction $R \mapsto \operatorname{Spec} R$ is functorial in R in a contravariant sense. That is, given a ring homomorphism $f : R \to S$, there is an induced map:*

$$F : \operatorname{Spec} S \to \operatorname{Spec} R,$$

defined by sending a prime ideal $\mathfrak{p} \subseteq S$ to its preimage $f^{-1}(\mathfrak{p}) \subseteq R$, which is easily verified to be a prime ideal. This establishes a contravariant functor

$$\operatorname{Spec} : \mathsf{CRing} \to \mathsf{Sets},$$

from the category of commutative rings with identity to the category of sets.

Example 5.4. The following is a list of basic examples of the spectrum of a ring:

(1) The spectrum of a ring is empty if and only if the ring is the zero ring.
(2) If $R = \mathbb{K}$ is a field, then the spectrum of $\mathbb{K}$ is a single point, (0). The corresponds to the notion of a point in affine algebraic geometry.
(3) Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Classical affine algebraic geometry can be reformulated as the study of the scheme $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$. More generally, for any ring R, we define

$$\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \ldots, x_n],$$

which is called affine n-space over R. When the base ring R is clear from context, we will simply write $\mathbb{A}^n$.
(4) Let $R = \mathbb{Z}$. We have

$$\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \{(p) \mid p \text{ prime}\}$$

Schemes over $\mathbb{Z}$, $\mathbb{Q}$, or more generally over number fields, play a central role in the application of scheme theory to number theory.

Remark 5.5. *Example 5.4(b) raises an important concern. There are many non-isomorphic fields; however, the spectrum of all fields is the same singleton set. We will see later that the structure sheaf will allow us to distinguish between non-isomorphic fields.*

We now endow the spectrum of a ring with a topology. In affine algebraic geometry, the Zariski topology is defined as the coarsest topology in which all affine algebraic sets are closed. The definition of the Zariski topology on a a spectrum of a ring (or an affine scheme) is motivated by its classical counterpart in affine algebraic geometry.

Definition 5.6. Let R be a ring. The **Zariski topology** on $\operatorname{Spec} R$ is given by declaring the closed sets to be of the form,

$$V(S) = \{\mathfrak{p} \in \operatorname{Spec} R \mid S \subseteq \mathfrak{p}\},$$

for all subsets S of R.

Remark 5.7. *Note that $V(S) = V(\langle S \rangle)$.*

Remark 5.8. *If $\mathbb{K}$ is an algebraically closed field and $R = \mathbb{K}[x_1, \cdots, x_n]$ the classical algebra-geometry correspondence implies that $V(S)$ can be identified with the set of all irreducible affine algebraic sets in $\mathbb{A}^n$ that are contained in the affine algebraic set $\mathbb{V}(S)$.*

Remark 5.9. *Here is another way to think of $V(S)$. For $\mathfrak{p} \in \operatorname{Spec} R$, define the residue field $\kappa(\mathfrak{p})$ to be the field of fractions of the integral domain $R/\mathfrak{p}$. For $f \in R$, note that we have*

$$V(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f(\mathfrak{p}) = 0 \text{ in } \kappa(\mathfrak{p})\}.$$

Here $f(\mathfrak{p}) := \overline{(f)}$ is the image of (f) in $R/\mathfrak{p}$. In other words, we can think of $f \in R$ as a function on $\operatorname{Spec} R$, and $f(\mathfrak{p})$ is the element $\overline{(f)}$ in $R/\mathfrak{p}$. Consider $R = \mathbb{C}[z]$. For $f \in \mathbb{C}[z]$, note $(\overline{f}) = 0$ in $R/(z-a)$ if and only if $f(a) = 0$. More precisely, the evaluation map

$$\begin{aligned} \mathrm{Ev}_a : \mathbb{C}[z] &\to \mathbb{C} \\ f &\mapsto f(a) \end{aligned}$$

induces an isomorphism $\mathbb{C}[z]/(z-a) \cong \mathbb{C}$. For instance, we have

$$V(z^2 + z + 1) = \{(z - e^{i\pi/3}), (z - e^{i2\pi/3})\}.$$

Let us verify that the Zariski topology on $\operatorname{Spec}(R)$ is indeed a topology. The argument is similar to that of Proposition 3.1, but in this setting we must work purely algebraically to check that the collection of closed sets satisfies the axioms of a topology.

Lemma 5.10. *Let R be a ring. The following statements are true:*

(1) $V(0) = \operatorname{Spec}(R), V(1) = \emptyset$.
(2) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
(3) If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $V(\sum_i \mathfrak{a}) = \bigcap_i V(\mathfrak{a}_i)$.
(4) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{a}}$.
(5) $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$

Proof. The proof is given below:

(1) This is clear.
(2) If $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, then $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Conversely, if $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$, and if $\mathfrak{b}$ is not contained in $\mathfrak{p}$, for example, then there is a $b \in \mathfrak{b}$ such that $b \notin \mathfrak{p}$. Now, for any $a \in \mathfrak{a}$, $ab \in \mathfrak{p}$, so we must have $a \in p$ since $\mathfrak{p}$ is a prime ideal. Thus, $\mathfrak{a} \subseteq \mathfrak{p}$.
(3) $\mathfrak{p}$ contains $\sum_i \mathfrak{a}_i$ if and only if $\mathfrak{p}$ contains each $\mathfrak{a}_i$, simply because $\sum_i \mathfrak{a}_i$ is the smallest ideal containing all of the ideals $\mathfrak{a}_i$.
(4) Recall that the radical of $\mathfrak{a}$ is the intersection can be defined as:

$$\sqrt{a} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p}.$$

This shows that

$$\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}} \iff V(\mathfrak{a}) \supseteq V(\mathfrak{b})$$

(5) This follows from the definition of $\sqrt{\mathfrak{a}}$.

This completes the proof. □

Lemma 5.10 shows that finite unions and arbitrary intersections of sets of the form $V(\mathfrak{a})$ are again of that form. Hence, these sets define the closed sets of a topology on $\operatorname{Spec}(R)$.

Example 5.11. Consider $\operatorname{Spec} \mathbb{Z}$. Let us describe the closed subsets. These are of the form $V(I)$ where $I \subseteq \mathbb{Z}$ is an ideal, so $I = (n)$ for some $n \in \mathbb{Z}$.

(1) If $n = 0$, the closed subset is all of Spec($\mathbb{Z}$).
(2) If $n \neq 0$, then n has finitely many prime divisors. Hence
$$V(n) = \{(p_1), \dots, (p_{k_n}) \mid p_i \text{ is prime and } p_i \mid n\}$$
For example, $V(33)$ is $\{(3), (11)\}$.

We have seen that Spec induces a contravariant functor from $\mathsf{CRings} \to \mathsf{Sets}$ (Remark 5.3). Next, we check that the morphisms induced on Spec's from a ring-homomorphism are in fact continuous maps of topological spaces.

Lemma 5.12. Spec *induces a contravariant functor*
$$\operatorname{Spec} : \mathsf{CRing} \to \mathsf{Top},$$
from the category of CRing *commutative rings to the category* Top *of topological spaces.*

Proof. Let $f : R \to S$ be a ring homomorphism. We claim that the induced map
$$F : \operatorname{Spec} S \to \operatorname{Spec} R,$$
$$\mathfrak{p} \mapsto f^{-1}(\mathfrak{p}).$$
is continuous in the Zariski topology. Consider a closed subset $V(I) \subseteq \operatorname{Spec} R$, where $I \subseteq R$ is an ideal. Then the preimage under F is given by
$$F^{-1}(V(I)) = \{\mathfrak{p} \in \operatorname{Spec} S \mid f^{-1}(\mathfrak{p}) \supseteq I\}.$$
This is precisely the set of prime ideals $\mathfrak{p} \subseteq S$ such that $\mathfrak{p} \supseteq f(I)$, which is the definition of the closed subset $V(f(I)) \subseteq \operatorname{Spec} S$[4]. Thus,
$$F^{-1}(V(I)) = V(f(I)) = V(\langle f(I) \rangle),$$
showing that F is continuous. □

Note that we can also characterize the Zariski topology in terms of open sets. Indeed, for any $f \in R$, define
$$U_f := \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}.$$
Then U_f is the subset of $\operatorname{Spec} R$ consisting of prime ideals that do not contain f. This is precisely the complement of the closed set $V(f)$, so U_f is open in the Zariski topology.

Lemma 5.13. *Let R be a ring. The sets U_f form a basis for the Zariski topology on* $\operatorname{Spec} R$.

Proof. Suppose $U \subseteq \operatorname{Spec} R$ is open. We have $U = V(I)^c$ for some ideal I. Note that we have
$$V(I) = \bigcap_{f \in I} V(f) \implies U = \bigcup_{f \in I} V^c(f) = \bigcup_{f \in I} U_f$$
This completes the proof. □

Example 5.14. Let $R = \operatorname{Spec} \mathbb{C}[z]$. For $f \in \mathbb{C}[z]$, the open set U_f consists of all prime ideals $(x - z)$ such that $f(z) \neq 0$. Therefore,
$$V(f) = U_f^c \cong \{z \in \mathbb{C} \mid f(z) = 0\}$$
If f is not the zero polynomial, then $V(f)$ is a finite subset of $\mathbb{C}$. Otherwise, $V(0) = \mathbb{C}$.

[4]For $\mathfrak{p} \in \operatorname{Spec} S$, we have $F(\mathfrak{p}) \in V(I)$ if and only if $f^{-1}(\mathfrak{p}) \supseteq I$, which holds if and only if $\mathfrak{p} \supseteq f(I)$, i.e., $\mathfrak{p} \in V(f(I))$.

Proposition 5.15. *Let R be a a ring. The collection of open sets U_f have the following properties:*

(1) $U_1 = \operatorname{Spec} R$ and $U_0 = \emptyset$
(2) $U_{fg} = U_f \cap U_g$.
(3) $U_f = \emptyset$ iff f is nilpotent.
(4) $U_f = \operatorname{Spec} R$ iff f is a unit.
(5) More generally, $\bigcup_{i\in I} U_{f_i} = \operatorname{Spec} R$ if and only if the ideal generated by $\{f_i\}_{i\in I}$ is R.
(6) $U_f \subseteq U_g$ iff $f \in \sqrt{\mathfrak{g}}$.

Proof. The proof is given below:

(1) This follows because because prime ideals are not allowed to contain the unit element and because every prime ideal contains 0.
(2) This follows because fg lies in a prime ideal $\mathfrak{p}$ if and only if one of f, g does.
(3) This follows because $U_f = \emptyset$ iff every prime ideal contains f iff f is in the intersection of all prime ideals, i.e., nilradical.
(4) This follows because $U_f = \operatorname{Spec} R$ iff no prime ideal contains f iff f is a unit.
(5) This follows because $\bigcup_{i\in I} U_{f_i} = \operatorname{Spec} R.$ if and only if the ideal generated by $\{f_i\}_{i\in I}$ is R. Note that $\bigcup_{i\in I} U_{f_i} = \operatorname{Spec} R$ iff that all prime ideals cannot contain all f_i's iff for every prime ideal, $\mathfrak{p}$, there exists a $i \in I$ such that $f_i \notin \mathfrak{p}$ iff no prime ideal contains the ideal generated by $\{f_i\}_{i\in I}$ iff the ideal generated by $\{f_i\}_{i\in I}$ is R.
(6) This follows directly from Lemma 5.10.

This completes the proof. □

Remark 5.16. *For basic open sets, we will use the notation $U_f, V_f, D(f)$ interchangeably*

Remark 5.17. *Let $f : R \to S$ be a ring homomorphism, and let $F : \operatorname{Spec} S \to \operatorname{Spec} R$ denote the induced morphism of affine schemes. Then, for any $r \in R$, we have the identity*

$$F^{-1}(U_r) = U_{f(r)},$$

Let us define an operation that is, in a certain sense, an "inverse" to the process of taking closed subsets in $\operatorname{Spec} R$. This construction is analogous to the definition of $\mathbb{I}$ in classical affine algebraic geometry, where one assigns to a subset of affine space the ideal of all polynomials vanishing on it. This perspective will allow us to formulate and prove an analogue of the algebra-geometry correspondence in the more general setting of spectrum of a ring (or an affine scheme).

Definition 5.18. Let R be a ring. Given a subset $S \subseteq \operatorname{Spec} R$, define

$$I(S) = \{f \in R \mid f \in \mathfrak{p} \text{ for all } \mathfrak{p} \in S\} = \bigcap_{\mathfrak{p}\in S} \mathfrak{p} \subseteq R$$

Example 5.19. Let $R = \operatorname{Spec} \mathbb{C}[z_1, \cdots, z_n]$. Then

$$I(S) = \bigcap_{(z-a)\in S} \{f \in \mathbb{C}[z_1, \cdots, z_n] \mid f(a) = 0\}$$

Identifying $S \subseteq \operatorname{Spec} \mathbb{C}[z_1, \cdots, z_n]$ with a subset of $\mathbb{C}^n$, $I(S)$ is the set of polynomials in $\mathbb{C}[z_1, \cdots, z_n]$ that vanish at all points of S. Hence, the operator $I(\cdot)$ generalizes the operator $\mathbb{I}(\cdot)$ discussed before.

Proposition 5.20. *The $I(\cdot)$ operation has the following properties:*

(1) $I(S)$ is an an ideal of R.
(2) $I(\cdot)$ is inclusion-reversing: if $S_1 \subseteq S_2$, then $I(S_2) \subseteq I(S_1)$.
(3) $I(S_1 \cup S_2) = I(S_1) \cap I(S_2)$
(4) $V(I(S)) = \overline{S}$.
(5) $I(V(J)) = \sqrt{J}$ for an ideal J in R.
(6) (**Nullstellensatz**) We have the following bijection:

$$\{\text{Closed sets in Spec}R\} \longleftrightarrow \{\text{Radical ideals in } R\}$$
$$S \mapsto I(S)$$
$$V(\mathfrak{a}) \leftarrow\!\shortmid \mathfrak{a}$$

Proof. (1), (2) and (3) are clear. For (4), note that $S \subseteq V(I(S))$, because if $\mathfrak{p} \in S$, then clearly

$$I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p} \subseteq \mathfrak{p},$$

and so $\mathfrak{p} \in V(I(S))$. Conversely, if $V(\mathfrak{a})$ is any closed subset containing S, then this means that for any $\mathfrak{b} \in S$, we have $\mathfrak{b} \in V(\mathfrak{a})$ and hence $\mathfrak{a} \subseteq \mathfrak{b}$. Taking the intersection over all $\mathfrak{b}$ in S, we see that $I(S) \supseteq \mathfrak{a}$. Since I reverses inclusions we get that $V(I(S)) \subseteq V(\mathfrak{a})$. This shows that $V(I(S))$ is the smallest closed set which contains S. (5) follows because

$$I\left(V(\mathfrak{a})\right) := \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p} = \sqrt{\mathfrak{a}}$$

(6) follows from (4) and (5). □

Proposition 5.20 is the analog of the classical algebra-geometry correspondence (Proposition 3.6). While the classical correspondence connects radical ideals in a polynomial ring with the geometry of their zero sets in affine space, Proposition 5.20 provides a similar bridge in a more general setting.

6. Properties of Zariski Topology

We now discuss various topological properties of the Zariski topology on the spectrum of a ring. These properties include closed points, compactness, characterization of irreducible closed subsets. Understanding these features is essential for grasping the geometric intuition behind the algebraic structure of the spectrum.

6.1. Closed Points.

Note that Proposition 5.20(5) characterizes closed sets in the Zariski topology. In particular, we can characterize closed *points* in the Zariski topology as well.

Proposition 6.1. *Let R be a ring and $\mathfrak{p}$ be a prime ideal. We have $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. In particular, closed points of Spec R correspond to the maximal ideals of R.*

Proof. Observe that $V(E)$ is a closed set that contains the point $\mathfrak{p}$ if and only if $E \subseteq \mathfrak{p}$. Hence, we have

$$\overline{\{\mathfrak{p}\}} = \bigcap_{E \subseteq \mathfrak{p}} V(E) = V\left(\bigcup_{E \subseteq \mathfrak{p}} E\right) = V(\mathfrak{p}).$$

Hence, $\{\mathfrak{p}\}$ is closed if and only if

$$\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$$

if and only if there doesn't exist any prime ideal $\mathfrak{q}$ of $\operatorname{Spec} R$ properly containing $\mathfrak{p}$ if and only if $\mathfrak{p}$ is a maximal ideal. $\square$

We have a bijection:

$$\{\text{Closed Points in Spec } R\} \longleftrightarrow \{\text{Maximal Ideals of } R\}$$

Proposition 6.1 illustrates that the Zariski topology on $\operatorname{Spec} R$ behaves quite differently from the Euclidean topology on manifolds. In particular, $\operatorname{Spec} R$ is not Hausdorff if R contains a prime ideal that is not a maximal ideal. Consider the following example:

Example 6.2. Let R be an integral domain. Then (0) is a prime ideal that is not a maximal ideal. Hence, (0) is not a closed set since $\overline{(0)} = R$. In particular, $\operatorname{Spec} R$ is not Hausdorff.

Example 6.2 motivates the concept of a generic point.

Definition 6.3. Let R be a ring, A point $\mathfrak{p} \in \operatorname{Spec} R$ is called a generic point if $\overline{\{\mathfrak{p}\}} = \operatorname{Spec} R$.

We conclude this section with a proof that $\operatorname{Spec} R$ is a T_0 space. Recall that a topological space, X, is T_0 if $x, y \in X$ are distinct points, then either there exists a neighborhood of x that does not contain y, or there exists a neighborhood of y that does not contain x.

Proposition 6.4. *Let R be a ring.* $\operatorname{Spec} R$ *is a T_0 space.*

Proof. Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ be distinct points. Then we have $\mathfrak{p} \neq \mathfrak{q}$ as ideals. WLOG assume that $\mathfrak{q} \not\subseteq \mathfrak{p}$. Then there exists an element $\alpha \in \mathfrak{q}$ such that $\alpha \notin \mathfrak{p}$. The distinguished open set U_α is then an open neighborhood of $\mathfrak{p}$ that does not contain $\mathfrak{q}$. $\square$

6.2. **Irreducible Closed Sets.** We have characterized the closed subsets of the spectrum of a ring, R. We now turn to characterizing the irreducible closed subsets of $\operatorname{Spec} R$. In analogy with the classical algebra-geometry correspondence, these irreducible closed subsets correspond precisely to the prime ideals of R.

Proposition 6.5. *Let R be a ring. A closed subset $Z \subseteq \operatorname{Spec} R$ is irreducible if and only if Z is of the form $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. First assume that $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. By Proposition 5.20(5), we have

$$Z = V(\mathfrak{p}) = V(I(\mathfrak{p})) = \overline{\{\mathfrak{p}\}}$$

If Z is reducible, then

$$\mathfrak{p} \in \overline{\{\mathfrak{p}\}} \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$$

Hence, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$ and since $V(\mathfrak{a}), V(\mathfrak{b})$ are closed sets we have $Z = \overline{\{\mathfrak{p}\}} \in V(\mathfrak{a})$ or $V(\mathfrak{b})$, contradicting that Z is reducible. Hence, Z is irreducible. Conversely, assume that Z is a closed irreducible set. Let

$$Z = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$$

for some ideal $\mathfrak{a} \in \operatorname{Spec} R$. It suffices to show that $\sqrt{\mathfrak{a}}$ is a prime ideal. Assume this is not the case. Then there exist $b, c \in R$ such that $bc \in \sqrt{\mathfrak{a}}$ but $b, c \notin \sqrt{\mathfrak{a}}$. If $\mathfrak{b} = (b)$ and $\mathfrak{c} = (c)$, then $\mathfrak{b}\mathfrak{c} \in \sqrt{\mathfrak{a}}$ but $\sqrt{\mathfrak{b}}, \sqrt{\mathfrak{c}} \notin \sqrt{\mathfrak{a}}$. We claim that $V(\sqrt{\mathfrak{a}}) \subseteq V(\mathfrak{b}) \cup V(\mathfrak{c})$. Note that

$$V(\sqrt{\mathfrak{a}}) \subseteq V(\mathfrak{b}) \cup V(\mathfrak{c}) \iff \sqrt{\mathfrak{a}} = I(V(\sqrt{\mathfrak{a}})) \supseteq I(V(\mathfrak{b}) \cup V(\mathfrak{c})) = I(V(\mathfrak{b})) \cap I(V(\mathfrak{c})) = \sqrt{\mathfrak{b}} \cap \sqrt{\mathfrak{c}}$$

Since $\mathfrak{b}$ and $\mathfrak{c}$ are principal ideals, we have

$$\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}} \cap \sqrt{\mathfrak{c}} \iff \sqrt{\mathfrak{a}} \supseteq \mathfrak{b} \cap \mathfrak{c}$$

The latter condition is clearly true. Hence, Z is reducible, a contradiction. □

We have a bijection:

$$\{\text{Closed irreducible Subsets of Spec } R\} \longleftrightarrow \{\text{Prime Ideals of } R\}$$

Corollary 6.6. *Let R be a ring.* $\operatorname{Spec} R$ *is irreducible if and only if the nilradical of R is a prime ideal.*

Proof. This follows from Proposition 6.5 and the fact that $\operatorname{Spec} R = V(0)$ and $\sqrt{0}$ is the nilradical of R. □

Example 6.7. Let R be an integral domain and let $ab \in \mathscr{N}(R)$. By definition of $\mathscr{N}(R)$, we have $a^n b^n = (ab)^n = 0$ for some $n \in \mathbb{N}$. Since R is an integral domain, $a^n = 0$ or $b^n = 0$ if and only if a or b is contained in $\mathscr{N}(R)$ if and only if $\mathscr{N}(R)$ is a prime ideal. Hence, $\operatorname{Spec} R$ is irreducible.

An irreducible component of a topological space is a maximal irreducible subset. Recall that any topological space can be written as a union of its irreducible components. We can also characterize irrecucible components of the spectrum of a ring.

Proposition 6.8. *Let R be a ring. The irreducible components of* $\operatorname{Spec} R$ *are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of R.*

Proof. A maximal irreducible subspace of $\operatorname{Spec} A$ must be closed, that is of the form $V(\mathfrak{a})$ for some ideal a of A. Since it is irreducible, we can assume the ideal to be prime, so it is $V(\mathfrak{p})$. Now $\mathfrak{p}$ is minimal because $V(\mathfrak{p})$ is maximal. □

6.3. **Compactness & Noetherian-ness.** We first show that the spectrum of a ring is always a compact topological space. This is a very convenient property to have. However, a general scheme need not be compact, as we shall see later.

Proposition 6.9. *Let R be a ring. Then* $\operatorname{Spec} R$ *is compact. More generally, U_f is compact for every $f \in R$.*

Proof. Assume that

$$U_1 = \operatorname{Spec} R = \bigcup_{i \in I} U_{h_i}$$

is a union of basic open sets. This is true if and only if

$$V(1) = \bigcap_{i \in I} V(h_i) = V\left(\sum_{i \in I} h_i\right).$$

By Lemma 5.10, this implies $1 \in \sqrt{\sum_i (h_i)}$, or $1 \in \sum_i (h_i)$ for some n. This means that 1 can be expressed as a finite sum $1 = \sum_i b_i h_i$, $b_i \in R$. Hence a finite subset of the h_i's will do. □

Example 6.10. $\operatorname{Spec} R$ can have non-empty non-compact open sets. For instance, take $R = \mathbb{K}[x_1, x_2, \cdots]$. Then

$$\bigcup_{n \geq 1} D(x_n) \subseteq \operatorname{Spec}(\mathbb{K}[x_1, x_2, \ldots])$$

is open but not quasi-compact. A similar argument shows that U_f is compact for every $f \in R$.

We now discuss an additional topological property of the spectrum of a ring. Recall that Noetherian rings form a special class of rings satisfying certain finiteness conditions. It is natural to define an analogous notion of Noetherian-ness at the level of topological spaces, yielding topological spaces that satisfy certain finiteness properties.

Definition 6.11. A topological space X is called **Noetherian** if it satisfies the descending chain condition for closed subsets. Any sequence

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$$

of closed subsets eventually stabilizes. That is, there is a $r \in \mathbb{N}$ such that

$$Z_r = Z_{r+1} = \cdots .$$

Example 6.12. $\mathbb{A}^n$ is a Noetherian topological space. Indeed, if

$$Y_1 \supseteq Y_2 \supseteq \dots$$

is a descending chain of closed subsets, then

$$\mathbb{I}(Y_1) \subseteq \mathbb{I}(Y_2) \subseteq \dots$$

is an ascending chain of ideals in $R = \mathbb{K}[x_1, \dots, x_n]$. Since R is a Noetherian ring, this chain of ideals is eventually stationary. But for each i, $Y_i = \mathbb{V}(\mathbb{I}(Y_i))$, so the chain Y_i is also stationary.

Noetherian topological spaces possess desirable properties, making them more tractable due to the finiteness conditions imposed by the Noetherian property. We illustrate this with the following example result:

Proposition 6.13. *Suppose X is a Noetherian topological space. Then every nonempty closed subset Z can be expressed uniquely as a finite union $Z = Z_1 \cup \cdots \cup Z_n$ of irreducible closed subsets, none contained in any other.*

Proof. Consider the collection, Σ, of closed subsets of X that cannot be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Assume that $\Sigma \neq \emptyset$. Since X is Noetherian, there is a minimal element in Σ. Call it Y. By construction, Y is not irreducible. So we can write $Y = Y' \cup Y''$ where Y' and Y'' are both proper closed subsets of Y. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can Y, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them. We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z_1' \cup Z_2' \cup \cdots \cup Z_s'$$

are two such representations. Then $Z_1' \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_r \cap Z_1')$. Now Z_1' is irreducible, so one of these is Z_1' itself, say (without loss of generality) $Z_1 \cap Z_1'$. Thus $Z_1' \subseteq Z_1$. Similarly, $Z_1 \subseteq Z_j'$ for some j; but because $Z_1' \subseteq Z_1 \subseteq Z_j'$, and Z_1' is contained in no other Z_i', we must have $j = 1$, and $Z_1' = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa, so they must be the same list. □

Example 6.14 (Hartshorne I.1.3)**.** Let $\mathbb{K}$ be a field, and let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. We have the following equalities of ideals in $\mathbb{K}[x, y, z]$:

$$\begin{aligned}(x^2 - yz,\ xz - x) &= (x^2 - yz,\ x) \cap (x^2 - yz,\ z - 1)\\ &= (x,\ yz) \cap (x^2 - y,\ z - 1)\\ &= (x,\ y) \cap (x,\ z) \cap (x^2 - y,\ z - 1).\end{aligned}$$

Therefore, Y is the union of three irreducible components: two of them are lines, and the third is a plane curve.

Proposition 6.15. *(*Hartshorne I.1.7) *The following statements are true:*

(1) *The following conditions are equivalent for a topological space X:*
  (i) *X is Noetherian.*
  (ii) *Every nonempty family of closed subsets has a minimal element.*
  (iii) *X satisfies the ascending chain condition for open subsets.*
  (iv) *Every nonempty family of open subsets has a maximal element.*

(2) *A Noetherian topological space is compact.*

(3) *Any subset of a Noetherian topological space is Noetherian in its induced topology.*

(4) *A Noetherian space which is also Hausdorff must be a finite set with the discrete topology.*

Proof. The proof has been commented out as it follows from standard results in point-set topology. □

We conclude by showing that if R is a Noetherian ring, then $\operatorname{Spec} R$ is a Noetherian topological space.

Proposition 6.16. *Let R be a Noetherian ring. Then* $\operatorname{Spec} R$ *is a Noetherian topological space.*

Proof. Suppose

$$V(I_1) \supseteq V(I_2) \supseteq \dots$$

is a descending sequence of closed subsets of $\operatorname{Spec} R$. Using Lemma 5.10, we have that,

$$\sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \dots,$$

is an ascending sequence of ideals in R. Since R is a Noetherian ring, this sequence stabilizes. That is, there exist a r such that $\sqrt{I_r} = \sqrt{I_{r+1}} = \cdots$. Since $V(I) = V(\sqrt{I})$ for any ideal I in R, we have that the descending sequence of closed subsets also stabilizers. Hence $\operatorname{Spec}(R)$ is Noetherian. □

Example 6.17. Let R be a Notherian ring. Then $R[x_1]$ is a Noetherian ring. This is the celebrated the Hilbert basis theorem (Proposition 27.6). Therefore, $R[x_1, \cdots, x_n]$ is also a Noetherian ring. Hence, $\operatorname{Spec} R[x_1, \cdots, x_n]$ is a Noetherian topological space. In particular, $\operatorname{Spec} \mathbb{K}[x_1, \cdots, x_n]$ is a Noetherian topological space for any field $\mathbb{K}$.

Remark 6.18. *If* $\operatorname{Spec} R$ *is a Noetherian topological space, R need not be Noetherian. One example is*

$$R = \mathbb{K}[x_1, x_2, x_3, \dots]/(x_1, x_2^2, x_3^3, \dots)$$

Then $\operatorname{Spec} R$ *has one point, so it is a Noetherian topological space. But T is not a Noetherian ring.*

6.4. **Conectedness.** We can now characterize when the spectrum of a ring is disconnected. To do so, we invoke properties of the structure sheaf that is naturally endowed on the spectrum of a ring (see Proposition 11.8). The structure sheaf itself will be introduced in detail later, when we discuss sheaves in Part 2.

Proposition 6.19. *Let R be a ring.* $\operatorname{Spec} R$ *is disconnected if and only if R is a direct product of rings. More generally, the following are equivalent:*

(a) $\operatorname{Spec} R$ *is disconnected.*
(b) *There exist nonzero elements $e_1, e_2 \in R$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, and $e_1 + e_2 = 1$ (these elements are called orthogonal idempotents).*
(c) *R is isomorphic to a direct product $R_1 \times R_2$ of two nonzero rings.*

Proof. The proof is given below:

(1) (a) implies (c): Assume $\operatorname{Spec} R$ is disconnected. Then $\operatorname{Spec} R$ can be written as a union of two disjoint clopen subsets. Let these subsets be $V(\mathfrak{a})$ and $V(\mathfrak{b})$ for some ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$. Noting that $V(\mathfrak{a}) = \operatorname{Spec}(R/\mathfrak{a})$ and $V(\mathfrak{b}) = \operatorname{Spec}(R/\mathfrak{b})$, we have:

$$\operatorname{Spec} R = V(\mathfrak{a}) \bigsqcup V(\mathfrak{b}) = \operatorname{Spec}(R/\mathfrak{a}) \bigsqcup \operatorname{Spec}(R/\mathfrak{b}) \cong \operatorname{Spec}(R/(\mathfrak{a} \times \mathfrak{b})). \tag{*}$$

Since we can recover the ring by taking global sections of the structure sheaf[5], we conclude that

$$R \cong R/\mathfrak{a} \times R/\mathfrak{b}.$$

(2) (c) implies (b): Simply take $e_1 = (1, 0)$ and $e_2 = (0, 1)$. It is straightforward to verify that the desired properties hold.

(3) (b) implies (a): Since $e_1 e_2 = 0$, for every prime ideal $\mathfrak{p} \subset R$, we must have either $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$. Hence, the closed subsets $V(e_1)$ and $V(e_2)$ cover $\operatorname{Spec} R$. Suppose a prime ideal $\mathfrak{p}$ lies in both $V(e_1)$ and $V(e_2)$. Then $e_1, e_2 \in \mathfrak{p}$, and so $1 = e_1 + e_2 \in \mathfrak{p}$, implying $\mathfrak{p} = R$, which contradicts the definition of a prime ideal. Therefore, $V(e_1) \cap V(e_2) = \varnothing$, and the cover $\operatorname{Spec} R = V(e_1) \cup V(e_2)$ is by two disjoint closed subsets. It follows that $\operatorname{Spec} R$ is disconnected.

This completes the proof. □

Remark 6.20. *The last equality in Equation (*) follows from the fact that the prime ideals of the product ring $R_1 \times R_2$ are precisely those of the form $\mathfrak{p}_1 \times R_2$ and $R_1 \times \mathfrak{p}_2$, where $\mathfrak{p}_i$ is a prime ideal of R_i for $i = 1, 2$.*

7. Examples

The purpose of this section is to present a collection of examples of spectra of various rings. These examples serve to illustrate the diversity of behaviors that can arise and will be revisited throughout the text as more theory is developed. We begin by examining examples of spectra of rings that recover classical affine algebraic geometry as special cases.

Example 7.1. Let $\mathbb{K}$ be an algebraically closed field.

(1) Let's describe $\operatorname{Spec} \mathbb{K}[x]$, which is called the affine line over $\mathbb{K}$. Since $\mathbb{K}$ is algebraically closed and $\mathbb{K}[x]$ is a PID, we have

$$\operatorname{Spec} \mathbb{K}[x] = \{(x - a) \mid a \in \mathbb{K}\} \cup \{0\}$$

[5]Structure sheaf will be discussed in the next section.

We can identify each $a \in \mathbb{C}$ with the prime ideal $(x-a)$. Thus, the non-zero prime ideals of $\operatorname{Spec} \mathbb{K}[x]$ correspond bijectively to the points of $\mathbb{K}$.

(2) Consider the following spectrum of a ring:

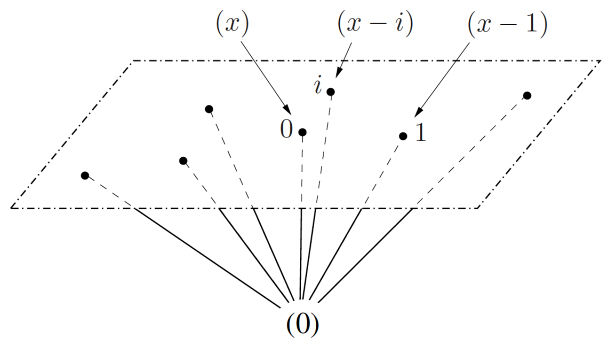
$$\operatorname{Spec} \mathbb{K}[x, y].$$

The ring $\mathbb{K}[x, y]$ is not a principal ideal domain, since the ideal (x, y) is not principal. Nevertheless, one can show that every prime ideal of $\mathbb{K}[x, y]$ is of one of the following forms:

$$(0), \quad (x-a, y-b), \quad (f),$$

where $a, b \in \mathbb{K}$, and $f \in \mathbb{K}[x, y]$ is an irreducible polynomial.

Remark 7.2. *Note that the zero ideal* (0) *in* $\mathbb{K}[x]$ *is contained in every non-zero prime (in this case, maximal) ideal.* (0) *is called the generic point of* $\mathbb{K}[x]$*. Generic point of* $\operatorname{Spec} R$ *will formally defined later.*



A picture of $\operatorname{Spec} \mathbb{C}[z]$. This picture is taken from [Vak17].

We can also examine the spectra of fields that are not algebraically closed. This provides one of the first indications that the language of schemes is well-suited for studying algebraic geometry over arbitrary fields, not just algebraically closed ones.

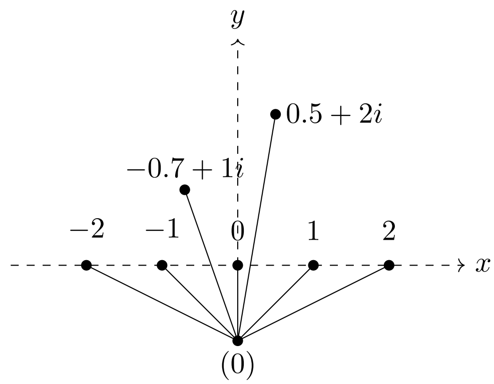
Example 7.3. Let $R = \operatorname{Spec} \mathbb{R}[x]$. We have

$$\operatorname{Spec}(\mathbb{R}[x]) = \{(0)\} \cup \{(x-a) \mid a \in \mathbb{R}\} \cup \{p(x)) \mid p(x) \text{ is an irreducible quadratic}\}$$

Note that (0) and $(x-a)$ for $a \in \mathbb{R}$ are all the maximal ideals of $\mathbb{R}[x]$. Maximal ideals of $\mathbb{R}[x]$ correspond to $\mathbb{R}$. Each $(p(x))$ can be identified with a complex number with positive imaginary part that is the root of $p(x)$.

Remark 7.4. *Example 7.3 highlights the key observation that, over a field that is not algebraically closed, there exist points on the affine line whose residue fields are nontrivial extensions of the base field. More precisely, if* $\mathbb{K}$ *is a field that is not algebraically closed, then there exist points* $x \in \mathbb{A}^1$ *such that the residue field is a proper field extension of* $\mathbb{K}$.

One might ask: why do we identify an irreducible quadratic $f(x) \in \mathbb{R}[x]$ with its complex root having positive imaginary part, rather than the conjugate root with negative imaginary part? This choice is a matter of convention, not mathematical necessity. The two complex conjugate roots correspond to the same prime ideal $(f(x))$ in $\operatorname{Spec} \mathbb{R}[x]$, since $f(x)$ is irreducible over $\mathbb{R}$. Therefore, selecting the root with positive imaginary part is simply a canonical way to represent the conjugate pair.



A picture of $\operatorname{Spec} \mathbb{R}[x]$

Remark 7.5. *Let $\mathbb{K}$ be a perfect field. Consider* $\operatorname{Spec} \mathbb{K}[x]$. *Since $\mathbb{K}$ is a field, $\mathbb{K}[x]$ is a PID. Therefore,*

$$\operatorname{Spec} \mathbb{K}[x] = \{f(x)) \mid p(x) \textit{ is an irreducible polynomial}\}$$

Since $\mathbb{K}$ is perfect, f has distinct roots in $\overline{\mathbb{K}}$[6] and these roots $r_1, \cdots, r_n$ form an orbit of $\overline{\mathbb{K}}$ under the action of the Galois group, G, of the field extension $\overline{\mathbb{K}}/\mathbb{K}$. Indeed, if any non-trivial subset of $S \subsetneq \{r_1, \cdots, r_n\}$ is G-invariant, then

$$\prod_{i \in S} (x - r_i)$$

would be an element of $\mathbb{K}[x]$ dividing f[7], contradicting that f is irreducible. Conversely, given any G-invariant finite subset $S \subsetneq \overline{\mathbb{K}}$ which has no non-trivial G-invariant subsets, the polynomial

$$\prod_{s \in S} (x - s)$$

is in $\mathbb{K}$ and irreducible by the same logic. So we have a bijection between $\overline{\mathbb{K}}/G$ and the non-zero prime ideals of $\mathbb{K}[x]$.

Let us now discuss the spectrum of quotient of a ring. If $I \subseteq R$ is an ideal, recall that there is a bijection:

$$\{\text{Prime ideals of } R/I\} \longleftrightarrow \{\text{Prime ideals of } R \,\text{containing}\, I\}$$

Thus we can picture $\operatorname{Spec} R/I$ as a subset of $\operatorname{Spec} R$. In fact, $\operatorname{Spec} R/I \cong V(I)$. In particular, $\operatorname{Spec} R/(f) \cong V(f)$ for $f \in R$.

Example 7.6. Consider the following examples:

(1) Consider $\mathbb{K}[x]/(x^2)$. An elementary argument shows that

$$\operatorname{Spec} \mathbb{K}[x]/(x^2) = \{(x)\}$$

[6]Because it is co-prime to its derivative.
[7]This follows from Galois theory.

(2) Consider $\mathbb{K}[x]/(x(x-1))$. Using the Chinese Remainder Theorem:
$$\mathbb{K}[x]/(x(x-1)) \cong \mathbb{K}[x]/(x) \times \mathbb{K}[x]/(x-1) \cong \mathbb{K} \times \mathbb{K}$$
Therefore, the spectrum of the ring is:
$$\operatorname{Spec} \mathbb{K}[x]/(x(x-1)) = \{(0) \times \mathbb{K},\ \mathbb{K} \times (0)\}$$

Remark 7.7. *The classical algebraic geometry correspondence implies that if $\mathbb{K}$ is an algebraically closed field , then points in $\operatorname{Spec}(\mathbb{K}/I)$ correspond to the points on $\mathbb{V}(I)$, as well as to the irreducible algebraic subsets of $\mathbb{V}(I)$. For example, points of $\operatorname{Spec} \mathbb{C}[x,y]/(xy)$ correspond to the points on the coordinate axes and the irreducible algebraic subsets of their union. One should use this intuition in the general case for a arbitrary commutative ring R.*

Let's now discuss the spectrum of a localized ring. Consider $\operatorname{Spec} S^{-1}R$, where S is a multiplicatively closed subset of R containing 1. Recall that there is a bijection between $\operatorname{Spec} S^{-1}R$ and the set of prime ideals $\mathfrak{p} \subseteq R$ such that $\mathfrak{p} \cap S = \emptyset$.

$$\{\text{Prime ideals of } S^{-1}R\} \longleftrightarrow \{\text{Prime ideals of } R \text{ that don't intersect with } S\}$$

Example 7.8. Let R be a ring. Consider the following examples:

(1) Let $S_f = \{1, f, f^2, \dots\}$. The prime ideals of $R_f := S_f^{-1}R$ correspond to the prime ideals of R that do not contain f. Hence,
$$\operatorname{Spec} R_f = \{\text{Prime ideals of } R \text{ that don't contain } f\}$$
Since $U_f \cap U_g = U_{fg}$, we have
$$\operatorname{Spec} R_f \cap \operatorname{Spec} R_g = \operatorname{Spec} R_{fg}$$

(2) Let $S_{\mathfrak{p}} = \mathfrak{p}^c$, where $\mathfrak{p}$ is a prime ideal. The prime ideals of $R_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}R$ are precisely the prime ideals of R that are contained in $\mathfrak{p}$. Hence,
$$\operatorname{Spec} R_{\mathfrak{p}} = \{\text{Prime ideals of } R \text{ contained in } \mathfrak{p}\}$$

Remark 7.9. *If $R = \mathbb{C}[x_1, \dots, x_n]$, the classical algebra-geometry correspondence implies that we can picture $\operatorname{Spec} S_f^{-1}\mathbb{C}[x_1, \dots, x_n]$ as the set of all points in $\mathbb{C}^n$ that do not lie on the zero set of f, along with irreducible affine algebraic sets not contained in the zero set of f.*

Local rings play a fundamental role in algebraic geometry, as they capture the behavior of schemes at a single point. The spectrum of a local ring provides a simple but illustrative example of a scheme with a distinguished closed point (corresponding to the maximal ideal) and a *generic* point. These examples are crucial for understanding local properties of schemes.

Example 7.10. Consider the localization of the polynomial ring $\mathbb{K}[x]$ at the prime ideal (x), denoted by $\mathbb{K}[x]_{(x)}$. Its spectrum, $\operatorname{Spec}(\mathbb{K}[x]_{(x)})$, consists of exactly two prime ideals:
$$\operatorname{Spec}(\mathbb{K}[x]_{(x)}) = \{(0), (x)\}$$
This provides an example of a scheme with a unique closed point (x) and a *generic point* (0).

Part 2. Sheaves

A *space*, such as a topological space or a smooth manifold, can often be studied through the algebra of functions defined on it. However, a generic space may admit few *globally defined* functions—for instance, consider a non-normal topological space or bounded holomorphic functions on $\mathbb{C}^n$. A more precise perspective is that the structure of a *space* can be understood by studying *locally defined* functions. This richer perspective is formalized using a mathematical object called a pre-sheaf. Pre-sheaves and sheaves are objects studied in sheaf theory. We will use sheaves to endow the spectrum of a ring with a collection of functions that generalize the polynomial and rational functions studied in classical affine algebraic geometry. More generally, sheaves can be used to define objects like vector bundles by specifying their spaces of sections over any open set and describing how those sections restrict to one another and glue together. Hence, sheaves are an important tool in algebraic geometry for keeping track of locally defined geometric data. By associating data—such as functions, sections, etc.—to open subsets of a space and ensuring compatibility across overlaps, sheaves enable a cohesive transition from local to global phenomena.

8. Definitions

Before we define sheaves, we first want to introduce the notion of pre-sheaves, which is simpler and yet very helpful in understanding sheaves. Philosophically, pre-sheaves provide a powerful framework for systematically organizing and locally defined data. More precisely, given a topological space, X, the idea of a pre-sheaf is to associate each open set in X with an object in a category, C in such a way that we can establish a map from a bigger open set to a smaller open set inside it. More formally, we have the following definition:

Definition 8.1. Let X be a topological space and let C be a category. Let $\mathsf{Open}(X)$ denote the category of open sets on X A C**-valued pre-sheaf** on X is a contravariant functor:

$$\mathscr{F} : \mathsf{Open}(X) \to \mathsf{C}$$

Remark 8.2. *If $\mathscr{F}$ is a pre-sheaf on X, we refer to $\mathscr{F}(U)$ as the sections of the pre-sheaf $\mathscr{F}$ over the open set U. We sometimes use the notation $\Gamma(U, \mathscr{F})$ to denote $\mathscr{F}(U)$. If $U \subseteq V$, we write $\rho_{V,U}$, $res_{V,U}$ or $|_U$ for the morphism between $\mathscr{F}(V)$ and $\mathscr{F}(U)$.*

Given a pre-sheaf on X, a natural question to ask is the extent to which its sections over an open set $U \subseteq X$ are determined by their restrictions to the open subsets of U. A sheaf is roughly speaking a pre-sheaf where the aforementioned question can be answered affirmatively.

Definition 8.3. Let X be a topological space and let C be a category admitting all limits[8]. A C**-valued sheaf** on X is a pre-sheaf if the following diagram is an equalizer for every open cover $\mathscr{U} = \{U_i\}_{i \in I}$ of any open set U:

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

The first map in Definition 8.3 is the product of the restriction maps

$$\mathrm{res}_{U,U_i} : \mathscr{F}(U) \to \mathscr{F}(U_i),$$

[8]It would be sufficient to require that C admits all products and equalizers. However, this assumption implies that C admits all limits.

and the pair of arrows are the products of the two sets of restrictions:

$$\mathrm{res}_{U_i, U_i \cap U_j} : \mathscr{F}(U_i) \to \mathscr{F}(U_i \cap U_j),$$
$$\mathrm{res}_{U_j, U_i \cap U_j} : \mathscr{F}(U_j) \to \mathscr{F}(U_i \cap U_j).$$

If $\mathsf{C} = \mathsf{Sets}, \mathsf{Ab}, R\text{-}\mathsf{Mod}$, the condition in the definition of a sheaf simplify to the conditions:

(1) (**Identity Axiom**) If $\{U_i\}_{i \in I}$ is an open cover of U, and $f_1, f_2 \in \mathscr{F}(U)$, and $f_1|_{U_i} = f_2|_{U_i}$ for all i, then $f_1 = f_2$.
(2) (**Gluing Axiom)** Suppose $\{U_i\}_{i \in I}$ is an open cover of U. Suppose for each i we have $f_i \in \mathscr{F}(U_i)$ such that $f_i = f_j$ in $\mathscr{F}(U_i \cap U_j)$. Then there is a unique $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$.

We can package the above argument in the form an equalizer diagram to reconstruct Definition 8.3. For each open cover $U = \bigcup_{i \in I} U_i$ of an open set $U \subseteq X$, there is a sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathscr{F}(U_i) \xrightarrow{\beta} \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j),$$

where the maps α and β are defined by the assignments

$$\alpha(s) = (s|_{U_i})_{i \in I},$$
$$\beta\left((s_i)_{i \in I}\right) = \left(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}\right)_{i,j \in I}.$$

Then $\mathscr{F}$ is a sheaf if and only if these sequences are exact. Indeed, exactness at $\mathscr{F}(U)$ means that α is injective, i.e., that $s|_{U_i} = 0$ for all $i \in I$ implies $s = 0$. Exactness in the middle means that $\ker \beta = \operatorname{Im} \alpha$; that is, elements $(s_i) \in \prod \mathscr{F}(U_i)$ satisfying

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \text{for all } i, j \in I$$

come from a global section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.

Remark 8.4. *We'll mostly be working with* $\mathsf{Ab}, R\text{-}\mathsf{Mod}$*,* CRing.

In some cases, it is assumed *a priori* that $\mathcal{F}(\emptyset)$ is a terminal object in C. For instance, if $\mathsf{C} = \mathsf{Ab}$, it is often assumed *a priori* that $\mathcal{F}(\emptyset) = 0$, where 0 denotes the trivial abelian group. However, we show that this is actually a consequence of the definition of a sheaf.

Lemma 8.5. *Let X be a topological space and let C be a category with all limits and a terminal object, T. If $\mathscr{F} : \mathsf{Open}(X) \to \mathsf{C}$ is a sheaf, then $\mathscr{F}(\emptyset) \cong T$.*

Proof. Let $U = \emptyset$. Since the product over objects in C indexed over $\emptyset$ is a terminal object, the equalizer condition becomes

$$\mathscr{F}(\emptyset) \xrightarrow{\;g\;} T \underset{\mathrm{Id}_T}{\overset{\mathrm{Id}_T}{\rightrightarrows}} T$$

The morphism $g : \mathscr{F}(\emptyset) \to T$ is unique since T is a terminal object. Let $X \in \mathsf{C}$ and let $f : X \to T$ be a unique morphism from X to T. The equalizer condition states that there exists a unique morphism $f' : X \to \mathscr{F}(\emptyset)$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathscr{F}(\emptyset) & \xrightarrow{\;g\;} & T & \underset{\mathrm{Id}_T}{\overset{\mathrm{Id}_T}{\rightrightarrows}} & T \\ & {\scriptstyle f'} \nwarrow & \uparrow {\scriptstyle f} & \nearrow {\scriptstyle f} & \\ & & X & & \end{array}$$

On the other hand, *any* morphism $f' : X \to \mathscr{F}(\emptyset)$ makes the left-most triangle commute. Therefore $\mathscr{F}(\emptyset)$ is an object such for any X there exists a unique morphism $X \to F(\emptyset)$. In other words, $F(\emptyset) \cong T$. □

Category theory teaches us to always define morphisms between mathematical objects. We now define morphisms of pre-sheaves, and similarly for sheaves. In other words, we will describe the category of pre-sheaves and the category of sheaves

Definition 8.6. Let X be a topological space and $\mathscr{F}$ and $\mathscr{G}$ be C-valued pre-sheaves. A **morphism of pre-sheaves**, $\varphi : \mathscr{F} \to \mathscr{G}$, is a natural transformation. That is, for each open set $U \subseteq X$ there exists a morphism from $\mathscr{F}(U) \to \mathscr{G}(U)$ such that whenever $U \subseteq V$, the following diagram

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \\ {\scriptstyle \mathrm{res}_{V,U}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ \mathscr{F}(U) & \xrightarrow[\varphi(U)]{} & \mathscr{G}(U) \end{array}$$

commutes.

Remark 8.7. *We denote by $\varphi(V) : \mathscr{F}(V) \to \mathscr{G}(V)$ the morphism on sections over an open set $V \subseteq X$, and we write its restriction to an open subset $U \subseteq V$ as*

$$\varphi(V)|_U : \mathscr{F}(V)|_U \to \mathscr{G}(V)|_U.$$

If C is a concrete category, and $s \in \mathscr{F}(V)$, the commutativity of the restriction diagram (written in componentwise notation) is expressed by the equation:

$$\varphi(s)|_U = \varphi_i(s|_U),$$

where $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ is the given morphism on a piece of an open cover $\{U_i\}_{i \in I}$, and $U \subseteq U_i \cap V$.

Remark 8.8. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups, R-modules etc. then we require all the maps $\varphi(U)$ to be homomorphisms in the appropriate category.*

Definition 8.6 makes the collection all pre-sheaves on X into a category, which we denote as $\mathsf{PreShv}(X, \mathsf{C})$. The category of sheaves on X, which we denote as $\mathsf{Shv}(X, \mathsf{C})$, is then a full subcategory of the category of pre-sheaves on X satisfying the identity and gluing axioms.

Example 8.9. Note the following basic examples:

(1) Consider the simplest case where $X = \{*\}$ is a one-point space. Then the category of presheaves $\mathsf{PreShv}(X, \mathsf{C})$ is equivalent to C itself.

(2) There is a natural forgetful functor

$$\mathsf{Shv}(X, \mathsf{C}) \to \mathsf{PreShv}(X, \mathsf{C}),$$

reflecting that every sheaf is in particular a presheaf.

9. Examples

Sheaves offer a unified language to study and solve problems that involve patching local solutions into a global context. This makes them essential tools in diverse fields, including algebraic geometry, topology, and complex analysis. Let's look at some examples of sheaves.

Example 9.1. Let $\mathsf{C} = \mathsf{Sets}$ and let $X = \{*\}$ be a one-point topological space. Lemma 8.5 implies that a C-valued sheaf on X is defined by $\mathscr{F}(*) = S$ and $\mathscr{F}(\emptyset) = \{\mathrm{pt}\}$, where $S \in \mathsf{Sets}$.

Sheaves of functions form an important example of sheaves. For instance, the assignment that sends each open set $U \subseteq X$ to the ring of functions on U defines a sheaf. These examples are central in areas such as differential geometry and complex analysis.

Example 9.2. The following is a list of some examples of sheaves of functions:

(1) If X is a topological space, the pre-sheaf of continuous functions, $\mathscr{C}$, defined by $U \mapsto \mathscr{C}(U)$, where $\mathscr{C}(U)$ is the abelian group of continuous functions on U (with usual restrictions), is a sheaf.
(2) If X is a topological space, the pre-sheaf of nowhere vanishing continuous functions, $\mathscr{C}^{\times}$, defined by $U \mapsto \mathscr{C}^{\times}(U)$, where $\mathscr{C}^{\times}(U)$ is the abelian group of no-where vanishing continuous functions on U (with usual restrictions), is a sheaf.
(3) If $X = \mathbb{C}^n$, the pre-sheaf of holomorphic functions, $\mathscr{O}$, defined by $U \mapsto \mathscr{H}(U)$, where $\mathscr{H}(U)$ is the abelian group of holomorphic functions on U (with usual restrictions), is a sheaf.
(4) If $X = \mathbb{C}^n$, the pre-sheaf of nowhere vanishing holomorphic functions, $\mathscr{O}^{\times}$, defined by $U \mapsto \mathscr{H}^{\times}(U)$, where $\mathscr{H}^{\times}(U)$ is the abelian group of non-where vanishing holomorphic functions on U (with usual restrictions), is a sheaf.

Note that $\mathscr{F}(\emptyset) = 0$[9] is forced by Lemma 8.5 in all examples above.

Remark 9.3. *We can easily generalize Example 9.2 by considering the sheaf of functions restricted to open subsets of the appropriate space. Moreover, all the examples discussed above are, in fact, examples of R-module valued sheaves with $R = \mathbb{R}, \mathbb{C}$ as appropriate.*

Arguably the most important example of a sheaf of functions in algebraic geometry is the sheaf of regular functions on an affine variety. This is the sheaf of functions that can be written as rational functions—that is, quotients of polynomials—that are regular on their domain of definition.

Example 9.4. (Sheaf of Regular Functions) Let $\mathbb{K}$ be an algebraically closed field, and let $X \subseteq \mathbb{A}^n$ be an affine variety. For an open set $U \subseteq X$, let $\mathscr{R}_X(U)$ be the ring of all rational functions which are regular on U:

$$\mathscr{R}_X(U) = \{f \in K(X) \,|\, f \text{ is regular on } U\}\,.$$

Note that if $V \subseteq U$, then $\mathscr{R}_X(U) \subseteq \mathscr{R}_X(V)$, so this defines a pre-sheaf by letting the restriction maps be the inclusion maps. The sheaf axioms are also satisfied.

(1) The identity axiom holds because if $f \in K(X)$ restricts to zero in some $\mathscr{R}_X(U)$, then this simply means that $f = 0$ in $K(X)$.
(2) The gluing axiom holds because if $\{f_i \in \mathscr{R}_X(U_i)\}$ is a collection of rational functions that agree on the overlaps $U_i \cap U_j$ of an open covering, then they are all equal to the same element $f \in K(X)$. This rational function f must in turn be regular on all of $U = \bigcup_i U_i$ because if $p \in U$, then p lies in some U_j, and hence $f = f_j$ can be written as a/b with $b(p) \neq 0$.

It follows that $\mathscr{R}_X$ defines a CRing-valued sheaf, called the sheaf of regular functions on X.

[9]Here 0 is the trivial abelian group.

The next examples concern constant pre-sheaves and sheaves, which, despite their simplicity, play a foundational role in the development of sheaf cohomology. They also provide key intuition for understanding how local data can fail to glue globally.

Example 9.5. Let X be a topological space and let $\mathsf{C} = \mathsf{Ab}$. Let $A \in \mathsf{Ab}$ with the discrete topology. The following are two examples of Ab-valued pre-sheaves:

(1) For any non-empty open set $U \in \mathsf{Open}(X)$, let $\overline{A}(U) = A$. Clearly, $\overline{A}$ is a pre-sheaf with restriction maps the identity. This is called the **constant pre-sheaf**.
(2) For any non-emptyset open set $U \in \mathsf{Open}(X)$, let $\underline{A}(U)$ be the abelian group of all continuous maps of U into A. Then with the usual restriction maps (as in the previous example), we obtain a sheaf. Note that each function in $\underline{A}(U)$ is locally constant for each open set of X. This is called the **constant sheaf**.

Once again, $\mathscr{F}(\emptyset) = 0$ is forced by Lemma 8.5.

Remark 9.6. *Let $\underline{A}$ be the constant sheaf. Note that for every connected open set U, $\underline{A}(U) \cong A$ since the image of a continuous map from a connected set to a discrete space is constant. This justifies the terminology.*

Example 9.7. All examples discussed in Example 9.2 are examples of sheaves of R-modules with $R = \mathbb{R}, \mathbb{C}$ as appropriate.

Example 9.8. Let X be a topological space and let 0 denote the trivial abelian group. Fix any abelian group, A, and $x \in X$. Consider the assignment

$$i_x^A(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

This is can be made into a pre-sheaf if for open sets $U, V \subseteq X$ such that $V \subseteq U$, the map $i_x^A(U) \to i_x^A(V)$ is defined such that:

(1) If $x \notin U$, then the map is simply the identity morphism $0 \to 0$
(2) If $x \in V$, then the map is simply the identity morphism $A \to A$
(3) If $x \in U \setminus V$, then the map is simply the unique morphism $A \to 0$.

It is easy to verify that this defines a pre-sheaf. Let $\{U_i\}_{i \in I}$ for an open cover for an open set $U \subseteq X$. The identity and gluing axioms are essentially satisfied since U contains x if and only if some U_i contains x. This is called the **skyscraper sheaf**.

Next, we consider the sheaf of sections, which is fundamental in relating sheaf theory to geometry by associating sheaves to vector bundles and other fibered structures.

Example 9.9. (Sheaf of Sections) Let X, Y be topological space and let $\pi : Y \to X$ be a continuous map. Recall that a section of π is a continuous map $\sigma : X \to Y$ such that $\pi \circ \sigma = \mathrm{Id}_X$. For an open non-empty set $U \subseteq X$, define $\mathscr{E}(U)$ to be the set of sections of π on U. That is,

$$\mathscr{E}(U) = \{\sigma : U \to Y \mid \sigma \text{ is continuous and } \pi \circ \sigma = \mathrm{Id}_U\}.$$

The empty set is sent to the singleton set and the restriction maps are are given by restriction of functions. This is called the pre-sheaf of sections of π. In fact, pre-sheaf of sections of π is a sheaf of sets. Indeed, since sections are indeed continuous function, it is clear that the identity axiom is satisfied. Similarly, the gluing axiom is also satisfied if we note that if $\{U_i\}_{i \in I}$ is an open cover of of U and $\sigma_i \in \mathscr{E}(U_i)$ such that $\sigma_i = \sigma_j \in \mathscr{E}(U_i \cap U_j)$, then the function $\sigma : U \to Y$ such that $\sigma|_{U_i} = \sigma_i$ is indeed a section.

Remark 9.10. *If Y is a topological group, the sheaf of sections is a sheaf of groups.*

Remark 9.11. *If $Y = X \times \mathbb{R}$, and π is projecting onto the first factor, then sections of π are just continuous maps $X \to \mathbb{R}$. In other words, the sheaf of sections generalizes the sheaf of real-valued continuous functions.*

Finally, we examine presheaves that fail to satisfy the sheaf axioms and thus are not sheaves. These examples illustrate the necessity of the gluing and locality conditions in the definition of a sheaf, and understanding such presheaves is crucial for constructing sheafifications and for deeper insights in sheaf cohomology.

Remark 9.12. *A pre-sheaf may not be a sheaf. Here are two examples:*

(1) *(Identity axiom fails) Let $X = \{*_1, *_2\}$ with the discrete topology. Let $\mathscr{F}$ be a pre-sheaf of abelian groups defined as follows:*

$$\mathscr{F}(\{*_1, *_2\}) = \mathbb{Z}, \quad \mathscr{F}(\{*_1\}) = \mathbb{Z}_2, \quad \mathscr{F}(\{*_2\}) = \mathbb{Z}_2, \quad \mathscr{F}(\emptyset) = 0,$$

with the obvious homomorphisms $\mathbb{Z} \to 0$ and $\mathbb{Z} \to \mathbb{Z}_2$ However, this is not a sheaf. Indeed, $X = \{*_1\} \cup \{*_2\}$. Then $2, 4 \in \mathbb{Z}$ such that these elements restrict to the 0 element in $\mathbb{Z}_2$. However, $2 \neq 4$.

(2) *(Gluing axiom fails) Let $X = \mathbb{R}$, and let $\mathscr{F}(U)$ be the abelian group of bounded functions on non-empty open sets U. Then $\mathscr{F}$ defines a pre-sheaf but not a sheaf. Indeed, let $X = \bigcup_{i \in \mathbb{Z}}^{\infty} (i, i+1]$ and let $f_i \equiv i$ on $(i, i+1]$. Since $V_i \cap V_j = \emptyset$ for $i \neq j$, trivially we have that $f_i = f_j$ on each $V_i \cap V_j = \emptyset$ for $i \neq j$. However, there is not $f \in \mathscr{F}(\mathbb{R})$ such that $f|_{(i,i+1]} = f_i$; otherwise, f must be an unbounded function.*

The constant pre-sheaf is usually not a sheaf. Let X be a topological space with two open sets, U_1, U_2, such that $U_1 \cap U_2 = \emptyset$. Let $A \in \mathsf{Ab}$ be a non-trivial abelian group and let $\overline{A}$ be the corresponding constant pre-sheaf on X. Specifically,

$$\overline{A}(U) = \begin{cases} A, & U \neq \emptyset, \\ 0, & U = \emptyset. \end{cases}$$

Let's show that the gluing axiom fails. Let $a_1 \neq a_2 \in A$ such that $a_i \in \overline{A}(U_i)$ for $i = 1, 2$. Let $U = U_1 \cup U_2$. The overlap condition is trivially satisfied. However, it is clear we cannot find any $a \in \overline{A}(U)$ such that

$$a|_{U_1} = a_1 \quad \text{and} \quad a|_{U_2} = a_2,$$

Hence, this constant pre-sheaf is not a sheaf. Conceptually, if we view elements of $\overline{A}(U)$ as constant functions $U \to A$, this example illustrates that we are attempting to take constant functions on U_1 and U_2 with different values and glue them to obtain a constant function on $U_1 \cup U_2$, which is impossible.

10. Stalks

Stalks are fundamental tools in sheaf theory, providing a way to study the behavior of a sheaf at a single point. This construction allows us to isolate and analyze local data while still capturing the global structure of the sheaf. Stalks play a crucial role in understanding the local-to-global correspondence in mathematics, as they bridge the gap between local properties (encoded in sections over open sets) and global phenomena. Let's first motivate the definition of a stalk with the help of an example.

Example 10.1. Let $X = \mathbb{C}^n$ and let $\mathscr{O}$ denotes the sheaf of holomorphic functions on X. For each $x \in X$ and open set U containing x, we define an equivalence relation on $\mathscr{O}(U)$

$$f \sim g \iff \text{there exists an open set } W \subseteq U \text{ containing } p \text{ such that } f|_W = g|_W$$

The equivalence class of a function $f \in \mathscr{O}(U)$ is called the germ of f at x and is denoted by $[f]_x$. The stalk of $\mathscr{O}$ at x, denoted $\mathscr{O}_x$, is the vector space of all germs of holomorphic functions at x. Addition and scalar multiplication of germs are defined by performing these operations on any representatives that are defined on the same open set. For example, addition is defined as:

$$[f]_x + [g]_x = [f + g]_x$$

Let's check that addition is well-defined. Assume that $[f]_x = [f']_x$ and $[g]_x = [g']_x$. Then there exist open sets $V, W \subseteq U$ such that $f|_V = f'|_V$ and $g|_W = g'|_W$. It is clear that on $V \cap W \subseteq U$, we have

$$f + g|_{V\cap W} = f' + g'|_{V\cap W}$$

This shows addition is well-defined. Similarly, it can be checked that scalar multiplication is well-defined.

Remark 10.2. *$\mathscr{O}_x$ is actually a ring. This can be checked easily. In fact, $\mathscr{O}_x$ is a local ring. Let $\mathfrak{m}_x \subseteq \mathscr{F}_x$ denotes germs vanishing at x. This certainly forms an ideal. In fact, the ideal is maximal since $\mathscr{F}_x/\mathfrak{m}_x \cong \mathbb{C}$. This is the unique maximal ideal since any germ not contained in $\mathfrak{m}_x$ is invertible.*

The construction given above can be applied to continuous or smooth functions on an appropriate space. We can now give the general definition of a stalk of a pre-sheaf, abstracting away from the previous example.

Definition 10.3. Let X be a topological space and let C be a category admitting filtered colimits. Let $\mathscr{F}$ be a C-valued pre-sheaf on X. The **stalk** of a pre-sheaf, $\mathscr{F}$, at a point $x \in X$, denoted by $\mathscr{F}_x$, is

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$$

Remark 10.4. *The stalk of a sheaf is the stalk of the underlying pre-sheaf.*

If $\mathsf{C} = R\text{-}\mathsf{Mod}$, then colimits exist in C and the characterization of colimit of a directed system allows us to unpack the definition of the stalk of a pre-sheaf. For example, let $\mathscr{F}$ be a R-Mod-valued pre-sheaf on a topological space X. For each $x \in X$, the collection of R-modules $\mathscr{F}(U)$, where U ranges over all open sets containing x, together with the restriction maps, forms a direct system with the relation $U \leq V$ if $U \supseteq V$. The intersection of two open sets containing p serves as a common upper bound. Definition 10.3 defines the stalk of $\mathscr{F}$ at x as the direct limit of this system.

Remark 10.5. *Let's recall the direct limit of a directed system of R-modules. Recall that a directed set $(I, \leq)$ is a non-empty set I with a binary relation, $\leq$, that is reflexive and transitive, and where every pair of elements has a common upper bound. A direct system of R-modules consists of a family $\{M_\alpha\}_{\alpha \in I}$ of R-modules indexed by a directed set I, along with R-module homomorphisms $f_{\alpha\beta} : M_\alpha \to M_\beta$ for $\alpha \leq \beta$, satisfying*

$$f_{\alpha\alpha} = \mathrm{Id}_{M_\alpha},\ \alpha \in I$$

$$f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma},\ \alpha \leq \beta \leq \gamma$$

The direct limit (or colimit in this case) is defined by defining an equivalence relation on $\coprod_{\alpha\in I} M_\alpha$ *such that*

$m_\alpha \sim m_\beta \iff$ *there exists some* $\gamma \in I$ *such tht* $\alpha, \beta \leq \gamma$ *and* $f_{\alpha\gamma}(m_\alpha) = f_{\beta\gamma}(m_\beta) \in M_\gamma$

The direct limit of the direct system is denoted as

$$\varinjlim_{\alpha\in I} M_\alpha = \left(\coprod_{\alpha\in I} M_\alpha\right) / \sim$$

It is easy to check that $\varinjlim_{\alpha\in I} M_\alpha$ *is R-module. For example, addition is defined by*

$$[m_\alpha] + [m_\beta] = [f_{\alpha\gamma}(m_\alpha) + f_{\beta\gamma}(m_\beta)],$$

where γ *is some upper bound for* α *and* β*. This can be checked to be well-defined because all maps* $f_{\alpha\beta}$ *are homomorphisms. Other operations are defined in a similar manner.*

Example 10.6. Let X, Y be topological space and let $\pi : Y \to X$ be a local homeomorphism. Let $\mathscr{E}$ be the pre-sheaf of sections. We show that for each $x \in X$, we have

$$\mathscr{E}_x \simeq E_x = \pi^{-1}(x)$$

For each $x \in X$ such that $x \in U$, define

$$\begin{aligned} \eta_U : \mathscr{E}(U) &\to E_x, \\ s &\mapsto s(x) \end{aligned}$$

We use the maps η_U to induce a map on the colimit:

$$\begin{aligned} \eta : \mathscr{E}_x = \varinjlim_{x\in U} \mathscr{E}(U) &\to E_x, \\ [s] &\mapsto s(x) \end{aligned}$$

We check that η is well-defined. Let $[s_1], [s_2] \in \mathscr{F}_x$ such that $[s_1] = [s_2]$ If s_1, s_2 are defined such that $s_1 : U_1 \to E$ and $s_2 : U_2 \to E$, then there is a neighborhood of $x \in W \subseteq U_1 \cap U_2$ such that $s_1|_W = s_2|_W$. In particular, $s_1(x) = s_2(x)$. Hence η is well-defined. We claim that η is a bijection. First, we show that η is surjective. Because π is a local homeomorphism, given $e \in E_x = \pi^{-1}(x)$, we can find open neighborhoods O_e of e and U_x of $x = \pi(e)$ such that

$$\pi|_{O_e} : O_e \to U_x$$

is a homeomorphism. Then

$$(\pi|_{O_e})^{-1} : U_x \to O_e$$

is a section of π, and

$$\eta((\pi|_{O_e})^{-1}) = (\pi|_{O_e})^{-1}(x) = (\pi|_{O_e})^{-1}(\pi(e)) = e.$$

Hence η is surjective. Now we prove that η is injective. Suppose $\eta[s_1] = \eta[s_2]$. Then $s_1(x) = s_2(x)$. By using properties of local homeomorphisms, we can check that there is an open neighborhood of x on which s_1, s_2 agree. That is, $[s_1] = [s_2]$. Thus, η is injective.

Example 10.7. Let $\mathscr{R}$ be the sheaf of regular functions on an affine variety. Then the stalk $\mathscr{R}_x$ is the local ring $\mathcal{O}_{X,x}$.

Category theory teaches us to focus on the properties of morphisms between objects rather than the objects themselves. Consequently, we infer the concept of the stalk of a morphism of sheaves from the following result:

Lemma 10.8. *Let X be a topological space. Let C be a category admitting filtered colimits and let $\mathscr{F}, \mathscr{G}$ be C-valued pre-sheaves on X. There is a functor*

$$\mathscr{S}_x : \mathsf{PreShv}(X, \mathsf{C}) \to \mathsf{C}$$

called the stalkification at x functor for each $x \in X$. In particular, if there is a morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is an induced morphism on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ for each $x \in X$.

Proof. (Sketch) The functor is defined by mapping $\mathscr{F}$ to its stalk $\mathscr{F}_x$ for each $x \in X$. If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then the morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is induced by the universal property of colimits. It is straightforward to verify that this construction defines a functor. □

Remark 10.9. *If $\mathsf{C} = R$-Mod, then the morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ of stalks can be described concretely. It is the morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ such that $\varphi_x([f]_x) = [\varphi(f)]_x$. Let's check that this is well-defined. Suppose $[f]_x = [f']_x$ such that $f \in \mathscr{F}(U)$ and $f' \in \mathscr{F}(U')$. Then there exists an open set $W \subseteq U \cap U'$ containing x such that $f|_W = f'|_W$. We have*

$$\mathscr{F}(f)|_W = \mathscr{F}(f|_W) = \mathscr{F}(f'|_W) = \mathscr{F}(f')|_W$$

Hence $[\mathscr{F}(f)]_x = [\mathscr{F}(f')]_x$. It is easy to check that $\mathscr{F}_x$ is a morphism in C.

Sheaves are an important tool for keeping track of locally defined data. Therefore, we expect that many properties of sheaves can be checked at the level of stalks. We discuss some properties of sheaves that can be determined by looking at the corresponding stalks. Here is a sample proposition when $\mathsf{C} = R-\mathsf{Mod}$.

Proposition 10.10. *Let X be a topological space and $\mathscr{F}$ be a sheaf of R-modules on X.*

(1) *(**Sections are determined by stalks**) For $U \in \mathsf{Open}(X)$, the natural map*

$$h : \mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$$

is a monomorphism. Equivalently, the natural map

$$h : \mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$$

is injective. That is, if $f, g \in \mathscr{F}(U)$ then $f = g$ if and only if $f_x = g_x$ for all $x \in U$.

(2) *(**Morphisms are determined by stalks**) Let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves such that $\varphi_x = \psi_x$ for each $x \in X$. Then $\varphi = \psi$.*

Proof. The proof is given below:

(1) The forward direction is trivial. Conversely, assume that $f_x = g_x$ for each $x \in U$. For a fixed $x_0 \in U$, $f_{x_0} = g_{x_0}$ if and only if there exists a neighborhood $x_0 \in U_{x_0} \subseteq U$ such that

$$f|_{U_{x_0}} = g|_{U_{x_0}}.$$

If we take all such neighborhoods U_x for all $x \in U$, we get an open cover for $U = \cup_{x \in U} U_x$, and by the definition of sheaves,

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}(U_x)$$

is injective. Hence, $f = g$.

(2) Consider the following diagram:

$$\begin{array}{ccc} \mathscr{F}(U) & \longrightarrow & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \prod_{x\in U}\mathscr{F}_x & \longrightarrow & \prod_{x\in U}\mathscr{G}_x \end{array}$$

The top map is either $\varphi(U)$ or $\psi(U)$ and the bottom map is the corresponding induced map on stalks at each p. Since the diagram commutes by assumption, the following diagram commutes:

$$\mathscr{F}(U) \underset{\psi(U)}{\overset{\varphi(U)}{\rightrightarrows}} \mathscr{G}(U) \longrightarrow \prod_{x\in x}\mathscr{G}_x$$

Since the second map is a monomorphism by (1), we have $\varphi(U) = \psi(U)$ for each open set U. Hence, $\varphi = \psi$.

This completes the proof. □

Remark 10.11. *Proposition 10.10 is false for general pre-sheaves. Let $X = \{*_1, *_2\}$ with the discrete topology. Let $\mathscr{F}$ be a pre-sheaf of abelian groups defined as follows:*

$$\mathscr{F}(\{*_1, *_2\}) = \mathbb{Z}, \quad \mathscr{F}(\{*_1\}) = \mathbb{Z}_2, \quad \mathscr{F}(\{*_2\}) = \mathbb{Z}_2, \quad \mathscr{F}(\emptyset) = 0,$$

with the obvious homomorphisms $\mathbb{Z} \to 0$ and $\mathbb{Z} \to \mathbb{Z}_2$. Note that $\mathscr{F}_{*_1}, \mathscr{F}_{*_2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(1) *Let $U = X$. Note that the map*

$$\mathbb{Z} = \mathscr{F}(U) \to \prod_{x\in U}\mathscr{F}_x \cong \mathbb{Z}_2^4$$

is clearly not injective. Hence, a section is not necessarily determined by stalks for a general pre-sheaf.

(2) *Let φ be a morphism of $\mathscr{F}$. Let φ be such that $\varphi(X) : \mathbb{Z} \to \mathbb{Z}$ is the identity map. The consistency conditions for φ to be a sheaf morphism implies that the maps $\mathbb{Z}_2 \to \mathbb{Z}_2$ are identity maps. However, if $\varphi(X)$ is changed to $\varphi'(X) : \mathbb{Z} \to \mathbb{Z}$ which is multiplication by n map where n is the odd, consistency conditions for φ' to be a sheaf morphism implies that the maps $\mathbb{Z}_2, \to \mathbb{Z}_2$ are identity maps. In either case, the induced maps on stalks are identity. This shows that morphisms are not necessarily determined by stalks for a general pre-sheaf.*

11. Structure Sheaf

We now look at an example of a sheaf, called the *structure sheaf*, that is important in algebraic geometry. We have endowed the spectrum of a ring with the structure of a topological space. For a commutative ring R, to view $\operatorname{Spec} R$ as a *geometric space*, we equip it with a sheaf analogous to the sheaf of regular functions on an affine variety (Example 9.4). The structure sheaf on $\operatorname{Spec} R$ is constructed by considering the "regular fnunctions" associated with the ring R.

11.1. **Sheaves Defined on a Basis.** We first need to look at the concept of sheaves defined on a basis. Let $\mathsf{C} = \mathsf{CRing}$. We can also take $\mathsf{C} = \mathsf{Ab}, R\text{-}\mathsf{Mod}$. Sheaves are defined with $\mathsf{Open}(X)$ as the domain category for a topological space X. Given a basis $\mathscr{B}$ for X, we can attempt to track the sheaf data at the level of open sets in $\mathscr{B}$.

Definition 11.1. Let X be a topological space with basis $\mathscr{B}$. A CRing-valued **$\mathscr{B}$-pre-sheaf** is a contravariant functor:

$$\mathscr{F} : \mathsf{Open}^{\mathscr{B}}(X) \to \mathsf{CRing}$$

Here $\mathsf{Open}^{\mathscr{B}}(X)$ is the category of open sets of X in $\mathscr{B}$. A CRing-valued sheaf $\mathscr{B}$ pre-sheaf is a **CRing-sheaf** if the following diagram is an equalizer for every open cover $\mathscr{U} = \{U_i\}_{i \in I} \subseteq \mathscr{B}$ of any open set $U \in \mathscr{B}$:

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{V \subseteq U_i \cap U_j V \in \mathscr{B}} \mathscr{F}(V)$$

Lemma 11.2. *Let X be a topological space with basis $\mathscr{B}$, and let $\mathscr{F}$ be a CRing-valued $\mathscr{B}$-sheaf. We have*

$$\mathscr{F}(U) \cong \varprojlim_{\substack{V \subseteq U \\ V \in \mathscr{B}}} \mathscr{F}(V).$$

Proof. An element of $\varprojlim \mathscr{F}(V)$ defines a section on each base open set V, and these sections are compatible with restriction. Thus, by the gluing axiom this collection corresponds to a unique section over U, since the base open sets $V \in \mathscr{B}$ clearly form a cover of U. This gives a unique map

$$\varprojlim_{\substack{V \subseteq U \\ V \in \mathscr{B}}} \mathscr{F}(V) \longrightarrow \mathscr{F}(U)$$

such that for every base open set $W \subseteq U$, the following diagram commutes:

$$\begin{array}{ccc} \varprojlim_{\substack{V \subseteq U \\ V \in \mathscr{B}}} \mathscr{F}(V) & \longrightarrow & \mathscr{F}(U) \\ & \searrow & \downarrow \\ & & \mathscr{F}(W) \end{array}$$

Since any object with morphisms to each $\mathscr{F}(V)$ factors uniquely through the limit, this implies that $\mathscr{F}(U)$ satisfies the same universal property. This concludes the proof. $\square$

The primary motivation for introducing $\mathscr{B}$-sheaves is encapsulated in the following proposition, which states that, as anticipated, any $\mathscr{B}$-presheaf can be extended to a presheaf on X by *approximating* open sets in X through open sets in the basis $\mathscr{B}$.

Proposition 11.3. *Let X be a topological space with basis $\mathscr{B}$. Every CRing-valued $\mathscr{B}$-sheaf $\mathscr{F}$ extends to a CRing-valued sheaf $\overline{\mathscr{F}}$ on X, which is unique up to isomorphism.*

The idea of proof of Proposition 11.3 is to define $\overline{\mathscr{F}}(U)$ as the set of all possible gluings of sections of $\mathscr{F}$ over open sets in $\mathscr{B}$ that cover U. More precisely, a section $s \in \overline{\mathscr{F}}(U)$ is given by a collection of sections

$$s_i \in \mathscr{F}(V_i)$$

for some open cover $\{V_i\}_{i \in I}$ of U with $V_i \in \mathscr{B}$, such that

$$s_i|_W = s_j|_W$$

for any $W \in \mathscr{B}$ with $W \subseteq V_i \cap V_j$. The only drawback is that each section depends on the choice of a covering $\{V_i\}$. To define the group $\overline{\mathscr{F}}(U)$ in a manner that is independent of any particular cover, we consider the *largest cover* of U, consisting of all open sets $V \in \mathscr{B}$ such that $V \subseteq U$.

PROOF. Let $U \subseteq X$ be any open subset, and let $\mathscr{B}_U \subseteq \mathscr{B}$ denote the collection of open sets in $\mathscr{B}$ contained in U. Using Lemma 11.2, define

$$\bar{\mathscr{F}}(U) := \varprojlim_{V \in \mathscr{B}_U}$$

An element of $\mathscr{F}(U)$ is therefore given by a compatible family of sections $s_V \in \mathscr{F}(V)$. Note that we have

$$\overline{\mathscr{F}}(U) \ := \left\{ (s_V) \in \prod_{V \in \mathscr{B}_U} \mathscr{F}(V) : s_V|_W = s_W, \text{ for all } W \subseteq V \text{ with } W, V \in \mathscr{B}_U \right\}.$$

Note that if $U_1 \subseteq U$, then $\mathscr{B}_{U_1} \subseteq \mathscr{B}_U$, and the projection maps induce restriction maps

$$\overline{\mathscr{F}}(U) \to \overline{\mathscr{F}}(U_1).$$

This makes $\mathscr{F}$ into a pre-sheaf. The sheaf axioms are easily verified. Moreover, if U is an open set in $\mathscr{B}$, there is a canonical isomorphism

$$\mathscr{F}(U) \xrightarrow{\sim} \overline{\mathscr{F}}(U),$$

sending a section $t \in \mathscr{F}(U)$ to the collection $(s_V)_{V \in \mathscr{B}_U}$ where $s_V = t|_V$. The inverse is given by the projection onto the "U-th component". □

11.2. **Structure Sheaf.** To motivate the definition of the structure sheaf, let us recall the case when $X \subseteq \mathbb{A}^n$ is an affine variety, and $A(X)$ denotes its coordinate ring. For a distinguished open subset $D(f) \subseteq X$, we have

$$\mathcal{O}_X(D(f)) = A(X)_f,$$

where $A(X)_f$ is the localization of $A(X)$ at the element f (Proposition 4.4). Moreover, if $D(g) \subseteq D(f)$, the corresponding restriction map

$$\mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(g))$$

is given by the canonical localization map $A(X)_f \to A(X)_g$. In fact, the sheaf $\mathcal{O}_X$ is completely determined by the values $\mathcal{O}_X(D(f)) = A_f$ for each $f \in A$. Indeed, we have

$$\mathcal{O}_X(U) = \bigcap_{D(f) \subseteq U} \mathcal{O}_X(D(f)).$$

Motivated by this discussion, we can define a pre-sheaf on a basis of open subsets of the spectrum of an arbitrary ring R.

Definition 11.4. Let R be a ring, and let $\mathscr{B}$ denote the basis of distinguished open subsets of $\operatorname{Spec} R$. The **$\mathscr{B}$-structured sheaf** is contravariant functor

$$\mathscr{S} : \mathsf{Open}^{\mathscr{B}}(\operatorname{Spec} R) \longrightarrow \mathsf{CRing}$$
$$U_f \longmapsto R_f$$

We now show that $\mathscr{S}$ is a $\mathscr{B}$-sheaf. The result is proved using an algebraic lemma, which is also stated below.

Proposition 11.5. *Let R be a ring, and let $\mathscr{B}$ denote the basis of distinguished open subsets of* $\operatorname{Spec} R$*.*

(1) *Let $g_1, \ldots, g_r \in R$ be elements generating the unit ideal. For any R-module M, the following sequence is exact:*

$$0 \to M \xrightarrow{\alpha} \bigoplus_{i=1}^{r} M_{g_i} \xrightarrow{\beta} \bigoplus_{i,j=1}^{r} M_{g_i g_j},$$

where the maps α and β are defined by

$$\alpha(s) = (s/1, \ldots, s/1),$$
$$\beta(s_1, \ldots, s_r)_{i,j} = s_i/1 - s_j/1.$$

(2) *$\mathscr{S}$ is a* CRing*-valued $\mathscr{B}$-sheaf.*

Proof. The proof is given below:

(1)

(2) Let U_f is a distinguished open subset of $\operatorname{Spec} R$. First assume that $\{U_{f_i}\}_{i=1}^r$ is a finite open cover of U_f. Applying (1) to the ring R_f and the module $M = R_f$, we obtain an exact sequence:

$$0 \longrightarrow R_f \xrightarrow{\alpha} \bigoplus_{i=1}^{r} R_{f_i} \xrightarrow{\beta} \bigoplus_{i,j=1}^{r} R_{f_i f_j}.$$

The sheaf axioms are verified by the exactness of the sequence constructed above. Now assume that $\{U_{f_i}\}_{i \in I}$ is a general open cover of U_f. Since U_f is compact, there exists a finite subset $J \subseteq I$ such that $\{U_{f_j}\}_{j \in J}$ forms a finite subcover of U_f. We check the two sheaf axioms:

(a) If $s \in R_f$ maps to zero in R_{f_i} for every $i \in I$, then in particular, it maps to zero in R_{f_i} for each $i \in J$. The argument above for the finite cover case implies that $s = 0$ in R_f.

(b) Let $s_i \in R_{f_i}$ be compatible elements for $i \in I$. That is

$$s_i/1 = s_j/1$$

$R_{f_i f_j}$ for all $i, j \in I$. The argument above provides a unique element $s \in R_f$ such that $s_i = s/1 \in R_{f_i}$ for all $i \in J$. We show that this element s also induces the sections s_i for all $i \in I$. Fix an index $\alpha \in I$. Consider the finite covering

$$\{U_{f_i}\}_{i \in J \cup \{\alpha\}}$$

of U_f. The argument above implies there exists an element $s' \in R_f$ such that $s'/1 = s_i$ in R_{f_i} for all $i \in J$ and $s'/1 = s_\alpha$ in R_{f_α}. Since both s and s' restrict to the same elements in R_{f_i} for all $i \in J$, uniqueness implies that $s = s'$ in R_f. Hence, $s/1 = s_\alpha$ in R_{f_α} as well.

Hence, $\mathscr{S}$ is a $\mathscr{B}$-sheaf.

This completes the proof. □

The discussion above shows that the assignment $U_f \mapsto R_f$ define a sheaf on the basis of distinguished open sets of $\operatorname{Spec} R$. We now extend this to a sheaf on the entire space, giving the precise definition of the structure sheaf.

Definition 11.6. Let R be a ring. The **structure sheaf**, $\mathscr{O}_{\operatorname{Spec} R}$, is the unique sheaf extending the $\mathscr{B}$-sheaf, $\mathscr{S}$. In particular, $\mathscr{O}_{\operatorname{Spec} R}$ is defined as:

$$\mathscr{O}_{\operatorname{Spec}(R)}(U) = \left\{ (s_i) \in \prod_{i \in I} R_{f_i} \;\middle|\; s_i = s_j \text{ in } R_{f_i f_j} \text{ for all } i, j \in I \right\},$$

Remark 11.7. *Proposition 11.3 implies that Definition 11.6 is well-defined.*

Proposition 11.8. *Let R be a ring.*

(1) $\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$.
(2) *For any $\mathfrak{p} \in \operatorname{Spec} R$, we have*

$$\mathscr{O}_{\operatorname{Spec} A, \mathfrak{p}} \cong R_{\mathfrak{p}}.$$

Proof. We defined $\mathscr{O}_{\operatorname{Spec} R}$ so that $\mathscr{O}_{\operatorname{Spec} R}(U_f) \cong R_f$ for every $f \in R$. Taking $f = 1$, we obtain

$$\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$$

For (2), since the distinguished open sets form a basis for the topology, the stalk of $\mathscr{O}_{\operatorname{Spec} R}$ at a point $\mathfrak{p} \in \operatorname{Spec} R$ can be computed as the direct limit

$$\mathscr{O}_{\operatorname{Spec} R, \mathfrak{p}} \cong \varinjlim_{\mathfrak{p} \in U_f} \mathscr{O}_{\operatorname{Spec} R}(U_f) \cong \varinjlim_{f \notin \mathfrak{p}} R_f.$$

We claim that the natural map $\varinjlim_{f \notin \mathfrak{p}} R_f \to R_{\mathfrak{p}}$ induced by the maps $R_f \to R_{\mathfrak{p}}$ for $f \notin \mathfrak{p}$ is an isomorphism.

(a) Any element $a/s \in R_{\mathfrak{p}}$, with $s \in \mathfrak{p}^c$, lies in the image of the canonical map $R_s \to R_{\mathfrak{p}}$. Hence, the map is surjective.
(b) Suppose an element $a/f^n \in R_f$ maps to zero in $R_{\mathfrak{p}}$. This means there exists $s \in \mathfrak{p}^c$ such that $as = 0$ in R. Then $a/f^n = 0$ in R_g, where $g = sf$. Hence, the element vanishes in the direct limit. Hence, the map is injective.

Hence, we have

$$\mathscr{O}_{\operatorname{Spec} R, \mathfrak{p}} \cong \varinjlim_{f \notin \mathfrak{p}} R_f \cong R_{\mathfrak{p}}$$

This completes the proof. □

Example 11.9. The structure sheaf carries essential algebraic information beyond the underlying topological space. Let $\mathbb{K}$ be an algebraically closed field. Then $\operatorname{Spec} \mathbb{K}$ consists of a single point, (0) (Example 5.4). However, we have

$$\mathscr{O}_{\operatorname{Spec}(\mathbb{K})}(\mathbb{K}) \cong \mathbb{K}$$

Hence, the structure sheaf distinguishes non-isomorphic algebraically closed fields whose spectra are homeomorphic as single points.

12. Gluing Sheaves

A central theme in modern geometry and topology is the principle of constructing global objects from compatible local data. In the theory of sheaves, this idea is made precise through the process of gluing. The ability to glue sheaves is fundamental not only to the construction of sheaves themselves but also to many deeper results, such as the formulation of sheaf cohomology and the development of schemes in algebraic geometry. We discuss how to glue morphisms of sheaves and sheaves.

Remark 12.1. *We work with sheaves taking values in a concrete category, such as* CRing *(the category of abelian groups) since sheaves defining schemes discussed later on take values in concrete categories. Arguments below can be adapted to apply to* C*-valued sheaves, where* C *is a locally small category, by using the Yoneda embedding to reduce to the case of set-valued sheaves.*

Gluing morphisms of sheaves is the easiest of case.

Proposition 12.2. *Let X be a topological space and $\{U_i\}_{i\in I}$ is an open cover of X. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X taking values in a concrete category such as* CRing*. Suppose we are given, for each $i \in I$, a morphism of sheaves $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ such that for all $i, j \in I$, the restrictions agree on overlaps:*

$$\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}.$$

Then there exists a unique morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ satisfying

$$\varphi|_{U_i} = \varphi_i$$

for all $i \in I$.

Proof. Let $V \subseteq X$ be an open set, and let $s \in \mathscr{F}(V)$. We define $\varphi(s) \in \mathscr{G}(V)$. The open sets $V_i := U_i \cap V$ form an open cover of V. For each $i \in I$, consider the section

$$\varphi_i(s|_{V_i}) \in \mathscr{G}(V_i).$$

We have

$$\varphi_i(s)|_{V_i\cap V_j} = \varphi_i(s|_{V_i\cap V_j}) = \varphi_j(s|_{V_i\cap V_j}) = \varphi_j(s)|_{V_i\cap V_j},$$

which shows that the sections $\varphi_i(s|_{V_i})$ agree on the overlaps $V_i \cap V_j$. By the gluing axiom for the $\mathscr{G}$, there exists a unique section $\varphi(s) \in \mathscr{G}(V)$ such that $\varphi(s)|_{V_i} = \varphi_i(s|_{V_i})$ for all $i \in I$. We define $\varphi(s)$ to be this glued section. By construction, $\varphi|_{U_i} = \varphi_i$ for all $i \in I$. For uniqueness, suppose φ and ψ are two morphisms of sheaves such that $\varphi|_{U_i} = \psi|_{U_i}$ for all $i \in I$. Let $V \subseteq X$ be an open set, and let $s \in \mathscr{F}(V)$. Then for each $i \in I$, we have

$$\varphi(s)|_{V_i} = \varphi_i(s|_{V_i}) = \psi_i(s|_{V_i}) = \psi(s)|_{V_i}.$$

Thus, $\varphi(s)$ and $\psi(s)$ agree on the open cover $\{V_i\}$ of V. By the identity axioms for the sheaf $\mathscr{G}$, it follows that $\varphi(s) = \psi(s)$. Hence, $\varphi = \psi$ as morphisms of sheaves. □

We now discuss how to glue sheaves. Suppose we are given a sheaf $\mathscr{F}_i$ on each open set U_i of an open cover $\{U_i\}_{i\in I}$ of a topological space X. The goal is to construct a global sheaf $\mathscr{F}$ on X such that $\mathscr{F}|_{U_i} \cong \mathscr{F}_i$ for each $i \in I$. A necessary condition for such a sheaf $\mathscr{F}$ to exist is that the local sheaves $\mathscr{F}_i$ must be isomorphic on the overlaps $U_i \cap U_j$. Moreover, by providing a collection of isomorphisms on the intersections together with compatibility on triple overlaps (i.e., satisfying the cocycle condition), this gluing data becomes not only necessary but also sufficient to construct such a global sheaf.

Proposition 12.3 (Hartshorne II.1.22)**.** *Let X be a topological space with open cover $\{U_i\}_{i\in I}$. Suppose we are given:*

1. *for each $i \in I$, a sheaf $\mathscr{F}_i$ on U_i taking values in a concrete category such as* CRing*,*
2. *for each pair $i, j \in I$, an isomorphism of sheaves*

$$\tau_{ji} : \mathscr{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{F}_j|_{U_{ij}},$$

where $U_{ij} := U_i \cap U_j$,

satisfying the following cocycle conditions:

(a) $\tau_{ii} = \mathrm{Id}_{\mathscr{F}_i}$ *for all* $i \in I$,
(b) $\tau_{ji} = \tau_{ij}^{-1}$ *for all* $i, j \in I$,
(c) $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$ *over* $U_{ijk} := U_i \cap U_j \cap U_k$ *for all* $i, j, k \in I$.

Then there exists a sheaf $\mathscr{F}$ *on* X *taking values in a concrete category such as* CRing*, together with isomorphisms*

$$\nu_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i,$$

such that for all $i, j \in I$*, the following compatibility condition holds over* U_{ij}*:*

$$\nu_j = \tau_{ji} \circ \nu_i|_{U_{ij}}.$$

Moreover, the sheaf $\mathscr{F}$*, together with the isomorphisms* $\{\nu_i\}$*, is unique up to unique isomorphism.*

Proof. A section of $\mathscr{F}$ over an open set $V \subseteq X$ is given by a collection of compatible sections $s_i \in \mathscr{F}_i(V_i)$, where $V_i := U_i \cap V$, satisfying the condition that for all $i, j \in I$, the identifications

$$\tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \quad \text{in } \mathscr{F}_j(V_{ij})$$

hold, where $V_{ij} := U_{ij} \cap V$. In particular, we have:

$$\mathscr{F}(V) := \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}_i(V_i) \;\middle|\; \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \text{ for all } i, j \in I \right\}.$$

The restriction maps are defined componentwise: if $W \subseteq V$, then the restriction map is

$$\mathscr{F}(V) \to \mathscr{F}(W)$$
$$(s_i)_{i \in I} \mapsto (s_i|_{W_i})_{i \in I}.$$

This is well-defined because the transition isomorphisms τ_{ji} are compatible with restrictions; that is,

$$\tau_{ji}(s_i|_{W_{ij}}) = s_j|_{W_{ij}} \quad \text{whenever} \quad \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}}.$$

We next check the two sheaf axioms.

(1) (Identity Axiom) Let $s = (s_i) \in \mathscr{F}(V)$ be a section, and suppose that $s|_{V_\alpha} = 0$ for every open set V_α in an open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of V. Then, for each i and α, we have

$$s_i|_{U_i \cap V_\alpha} = 0 \quad \text{in } \mathscr{F}_i(U_i \cap V_\alpha).$$

Since the sets $\{U_i \cap V_\alpha\}_{\alpha \in \Lambda}$ form an open cover of $U_i \cap V$, and each $\mathscr{F}_i$ is a sheaf on U_i, it follows that

$$s_i = 0 \quad \text{in } \mathscr{F}_i(U_i \cap V).$$

As this holds for every i, we conclude that

$$s = 0 \quad \text{in } \mathscr{F}(V).$$

(2) (Gluing Axiom) Let $\{s_\alpha\}$, with $s_\alpha \in \mathscr{F}(V_\alpha)$, be a compatible family of sections over an open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of V. Compatibility means that for all α, β,

$$s_\alpha|_{V_{\alpha\beta}} = s_\beta|_{V_{\alpha\beta}},$$

where $V_{\alpha\beta} := V_\alpha \cap V_\beta$. Fixing $i \in I$, this induces a compatible family of sections

$$s_{\alpha,i} := s_\alpha|_{U_i \cap V_\alpha} \in \mathscr{F}_i(U_i \cap V_\alpha).$$

Since each $\mathscr{F}_i$ is a sheaf on U_i, the sections $\{s_{\alpha,i}\}_\alpha$ glue uniquely to a section

$$s_i \in \mathscr{F}_i(U_i \cap V).$$

By construction, the compatibility condition on the transition maps holds:

$$\tau_{ij}(s_j)|_{U_{ij}\cap V} = s_i|_{U_{ij}\cap V},$$

since this equality holds on each $V_\alpha \cap U_{ij}$ and the s_α are compatible. Hence, the tuple $s = (s_i)$ defines an element of $\mathscr{F}(V)$, which by construction restricts to s_α on each V_α.

We now construct isomorphisms

$$\nu_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i.$$

To avoid getting confused by the indices, we fix the index $0 \in I$. Suppose $V \subseteq U_0$ is an open set. The projection map

$$\nu_{0,V} : \mathscr{F}(V) \to \mathscr{F}_0(V),$$
$$(s_i)_{i\in I} \mapsto s_0.$$

induces a well-defined sheaf morphism

$$\nu_0 : \mathscr{F}|_{U_0} \to \mathscr{F}_0.$$

We claim that ν_0 is an isomorphism. Note that we have $\nu_\beta \circ \tau_{\beta 0} \circ \nu_0$. Indeed for $V \subseteq U_{0\beta}$ we have

$$\tau_{\beta 0}\big(\nu_0(s)\big) = \tau_{\beta 0}(s_0) = s_\beta = \nu_\beta(s).$$

We first show that ν_0 is injective. If $s = (s_i) \in \prod_{i\in I} \mathscr{F}_i(V)$ is a section such that $s_0 = 0 \in \mathscr{F}_0(V)$, then for all $i \in I$ we have

$$s_i = s_i|_{V\cap U_0} = \tau_{i0}(s_0) = 0,$$

and hence $s = 0$. We now show that ν_0 is surjective. Take any section $\sigma \in \mathscr{F}_0(V)$ over some $V \subseteq U_0$ and define

$$s := \big(\tau_{i0}(\sigma|_{V\cap U_i\cap U_0})\big)_{i\in I}.$$

Note that for every $i, j \in I$, we have

$$\tau_{ji}\big(\tau_{i0}(\sigma|_{V\cap U_i\cap U_j\cap U_0})\big) = \tau_{j0}(\sigma|_{V\cap U_i\cap U_j\cap U_0}).$$

Therefore, s defines an element of $\mathscr{F}(V)$. As $\tau_{00}(\sigma|_{V\cap U_0\cap U_0}) = \sigma$ by the first gluing condition, we also have $\nu_0(s) = \sigma$. This shows that ν_0 is an isomorphism. The proof of uniqueness is skipped. □

13. Pushword & Pullback Functors

Our discussion of sheaves has so far been confined to a single topological space X. However, it becomes essential to understand how sheaves defined on different topological spaces relate to one another in scheme theory. We now discuss the pushforward and pullback functors that allow us to related sheaves defined on different topological spaces

Remark 13.1. *We assume all sheaves are sheaves of R-modules.*

13.1. **Pushforward Functor.** Given a continuous map between topological spaces, we can ask how the sheaves on X and Y are related. We discuss the pushforward construction in this section. This notion will play a crucial role in the definition of (locally) ringed spaces in Section 14, which combines the notion of the spectrum of a ring with its associated structure sheaf.

Definition 13.2. Let $X, Y \in \mathsf{Top}$ and $f : X \to Y$ be a continuous map. If $\mathscr{F}$ is a sheaf on X the **pushforward sheaf**, $f_*\mathscr{F}$, on Y defined by

$$(f_*\mathscr{F})(U) := \mathscr{F}(f^{-1}(U)),$$

for every open set $U \subseteq Y$ and the restriction maps

$$(f_*\mathscr{F})(U) \to (f_*\mathscr{F})(V)$$

are defined to be those induced by the restriction maps of $\mathscr{F}$, for open sets $V \subseteq U \subseteq Y$.

Let's check that the direct image pre-sheaf is in fact a pre-sheaf. We begin with a simple observation. Since f is continuous, the preimage of any open set in Y is still an open set in X, and the operation of taking preimages preserves inclusions. Consequently, f naturally induces a functor f^{-1} from $\mathsf{Open}(Y)$ to $\mathsf{Open}(X)$. Moreover, by reversing the arrows in both categories, f^{-1} retains its functoriality. Now, let $\mathscr{F}$ be a pre-sheaf of R-modules over X, and consider the following diagram:

$$f_*(\mathscr{G}) : \mathsf{Open}^{\mathsf{op}}(Y) \xrightarrow{f^{-1}} \mathsf{Open}^{\mathsf{op}}(X) \xrightarrow{A} R\text{-}\mathsf{Mod}.$$

This gives the desired contravariant functor. Hence, the direct image pre-sheaf if indeed a pre-sheaf. If $\mathscr{F}$ is a sheaf, it is clear that the direct image pre-seaf is in fact a sheaf.

Example 13.3. The following is a basic list of computations of the pushforward.

(1) Let $i : \{x\} \hookrightarrow X$ be the inclusion of a closed point x into X, and let A be an abelian group, regarded as a constant sheaf on $\{x\}$. The pushforward sheaf is the skyscraper sheaf (Example 9.8):

$$i_*(A)(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let $i : U \hookrightarrow X$ be the inclusion of an open set U into X. If $\mathscr{F}$ is a sheaf on U, we have

$$i_*(\mathscr{F})(V) = \mathscr{F}(V \cap U)$$

(3) Let $\pi : X \hookrightarrow \{*\}$ be the map from X to a one point topological space. If $\mathscr{F}$ is a sheaf on X, we have

$$\pi_*(\mathscr{F})(*) = \mathscr{F}(X) = \Gamma(\mathscr{F})$$

Hence, π_* computes the global sections.

We can now define a functor:

$$f_* : \mathsf{PreShv}(X, R\text{-}\mathsf{Mod}) \to \mathsf{PreShv}(Y, R\text{-}\mathsf{Mod})$$

Let's verify that f_* is indeed a functor. If $\varphi : \mathscr{F} \to \mathscr{F}'$ is a morphism of pre-sheaves on X, we have the following commutative diagrams for open sets $U \subseteq V \subseteq Y$:

$$\begin{array}{ccc} f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)) & \longleftarrow & \mathscr{F}(f^{-1}(V)) = f_*\mathscr{F}(V) \\ \varphi_{f^{-1}(U)}\big\downarrow & & \big\downarrow\varphi_{f^{-1}(V)} \\ f_*\mathscr{F}'(U) = \mathscr{F}'(f^{-1}(U)) & \longleftarrow & \mathscr{F}'(f^{-1}(V)) = f_*\mathscr{F}'(V) \end{array}$$

Thus, $\varphi_{f^{-1}(-)}$ defines a pre-sheaf morphism $\varphi_{f^{-1}(-)} : f_*\mathscr{F} \to f_*\mathscr{F}'$. Hence, f_* is indeed a functor called the direct image functor. f_* also restricts to a functor

$$f_*|_{\mathsf{Shv}} : \mathsf{Shv}(X, R\text{-}\mathsf{Mod}) \to \mathsf{Shv}(Y, R\text{-}\mathsf{Mod})$$

This illustrates the functorial nature of sheaf-theoretic constructions—something we will explore further in later sections.

13.2. **Inverse Image Functor.** We now discuss the inverse image functor. Let $f : X \to Y$ be a continuous map between topological spaces, and let $\mathscr{G}$ be a sheaf on Y. Our goal is to determine whether it is possible to induce a pre-sheaf on X via f. Since the continuous image of an open set in X need not be open in Y, there is no immediate way to assign an R-module to each open set of X simply by composing with f. Nevertheless, we can emulate the construction of stalks by taking a colimit over all open neighborhoods in Y containing the image of a given open set in X.

Definition 13.4. Let X, Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $\mathscr{G}$ be a pre-sheaf on Y. The **inverse image pre-sheaf** $f^*\mathscr{G}$ on X is defined by the assignment

$$(f^*\mathscr{G})(U) = \varinjlim_{W \supseteq f(U)} \mathscr{G}(W)$$

where $U \subseteq X$ is any open set, and the colimit is taken over all open sets $W \subseteq Y$ containing $f(U)$.

Let's verify that $f^*\mathscr{G}$ is a pre-sheaf. Notice that if $U \subseteq V \subseteq X$ are open sets of X, then containing $f(V)$ will automatically contain $f(U)$. Thus, we obtain a natural restriction map from the universal property of the direct limit:

$$\varinjlim_{W \supseteq f(V)} \mathscr{G}(W) \to \varinjlim_{W \supseteq f(U)} \mathscr{G}(W).$$

Hence, $f^*\mathscr{G}$ is a pre-sheaf.

Example 13.5. Let Y be a topological space and let $\mathscr{G}$ is a pre-sheaf of R-modules on Y. We compute the inverse image pre-sheaf in some basic cases:

(1) Let $W \subseteq Y$ be an open set and let $f : W \to Y$ be the inclusion map. Let $U \subseteq W$ be an open set. Note that $U = U' \cap W$ where $U' \subseteq Y$ is an open set. We have

$$(f^*\mathscr{G})(U) = \varinjlim_{V \supseteq U' \cap W} \mathscr{G}(V) = \mathscr{G}(U' \cap W)$$

Hence, $f^*\mathscr{G} = \mathscr{G}|_W$

(2) Let $X = \{*\}$ and let $f : \{*\} \to Y$ be a continuous function such that $f(*) = y$. Note that we have

$$(f^*\mathscr{G})(\emptyset) = \varinjlim_{y \in \emptyset} \mathscr{G}(W) = \emptyset,$$
$$(f^*\mathscr{G})(W) = \varinjlim_{y \in W} \mathscr{G}(W) = \mathscr{G}_y.$$

Hence $f^*\mathscr{G} = \mathscr{G}_y$. Therefore stalks are just a special kind of inverse image pre-sheaf.

We can now define a functor:

$$f^* : \mathsf{PreShv}(Y, R\text{-}\mathsf{Mod}) \to \mathsf{PreShv}(X, R\text{-}\mathsf{Mod}).$$

Let's verify that f^* is indeed a functor. If $\varphi : \mathscr{G} \to \mathscr{G}'$ is a morphism of pre-sheaves on Y, m the universal property of colimits yields the following diagram:

$$\begin{array}{ccc} \mathscr{G}(W) & \xrightarrow{\varphi(W)} & \mathscr{G}'(W) \\ \downarrow & & \downarrow \\ \varinjlim_{W \supseteq f(U)} \mathscr{G}(W) & \dashrightarrow & \varinjlim_{W \supseteq f(U)} \mathscr{G}'(W) \end{array}$$

The functorality axioms are not hard to verify. Hence, f^* is indeed a functor.

Example 13.6. The inverse image pre-sheaf may not be a sheaf. Take $X = \{*_1, *_2\}$ with the discrete topology and let $Y = \{*\}$. Let $f : X \to Y$ be the constant map. Let $\mathscr{G}(\{*\}) = A$ be a non-trivial abelian group. Clearly, $\mathscr{G}$ is a sheaf of abelian groups on Y. We have

$$(f^*\mathscr{G})(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V) = A$$

for each open set $U \subseteq X$. Hence, $f^*\mathscr{G} = \overline{A}$ is the constant pre-sheaf which is not a sheaf.

Definition 13.7. Let X, Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $\mathscr{G}$ be a sheaf of R-modules on Y. The **inverse image sheaf** is the sheafification of the inverse image pre-sheaf.

Remark 13.8. *Note that we have a functor*

$$\mathsf{Shv}(Y, R\text{-}\mathsf{Mod}) \xrightarrow{f^*} \mathsf{PreShv}(X, R\text{-}\mathsf{Mod}) \xrightarrow{\mathrm{Sh}} \mathsf{Shv}(X, R\text{-}\mathsf{Mod})$$

By a abuse of notation, we continue to denote this functor by f^*.

We have introduced two induced sheaf functors f_* and f^* for any continuous map $f : X \to Y$. Since these two functors operate forward and backward between $\mathsf{Shv}(X, R\text{-}\mathsf{Mod})$ and $\mathsf{Shv}(Y, R\text{-}\mathsf{Mod})$, one might conjecture that they form an adjoint pair. This is indeed the case.

Proposition 13.9. (Hartshorne II.1.18) *The inverse image functor is left adjoint to the direct image functor. That is, for sheaves $\mathscr{F}$ and $\mathscr{G}$ on X and Y respectively, there is a bijection*

$$\mathrm{Hom}_{\mathsf{Shv}(X, R\text{-}\mathsf{Mod})}(f^*\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_{\mathsf{Shv}(Y, R\text{-}\mathsf{Mod})}(\mathscr{G}, f_*\mathscr{F})$$

Proof. We can exploit the adjunction between sheafification and forgetful functors to assume WLOG that f^* is a functor between pre-sheaf categories. In other word, it suffices to prove there is a bijection of sets

$$\mathrm{Hom}_{\mathsf{PreShv}(X,R\text{-}\mathsf{Mod})}(f^*\mathscr{G},\mathscr{F}) = \mathrm{Hom}_{\mathsf{Shv}(Y,R\text{-}\mathsf{Mod})}(\mathscr{G}, f_*\mathscr{F}).$$

Let $\varphi \in \mathrm{Hom}_{\mathsf{Shv}(Y,R\text{-}\mathsf{Mod})}(\mathscr{G}, f_*\mathscr{F})$. Fix any open set $U \subseteq X$ and let $W \subseteq Y$ be an open set such that $f(U) \subseteq W$. We have a morphism $\varphi(W) : \mathscr{G}(W) \to \mathscr{F}(f^{-1}(W))$. By the universal property of colimits, we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{G}(W) & \xrightarrow{\varphi(W)} & \mathscr{F}(f^{-1}(W)) \\ \big\downarrow & & \big\downarrow \\ \varinjlim_{W \supseteq f(U)} \mathscr{G}(W) & \underset{\beta_1(\varphi)(U)}{\dashrightarrow} & \mathscr{F}(U) \end{array}$$

It is not hard to see that $\beta_1(\varphi)(U)$ is compatible with restriction maps in pre-sheaves over X. Therefore, $\beta_1(\varphi)$ is a pre-sheaf morphism from $f_*\mathscr{F}$ to $\mathscr{G}$. This defines the map β_1:

$$\beta_1 : \mathrm{Hom}_{\mathsf{Shv}(Y,R\text{-}\mathsf{Mod})}(\mathscr{G}, f_*\mathscr{F}) \to \mathrm{Hom}_{\mathsf{PreShv}(X,R\text{-}\mathsf{Mod})}(f^*\mathscr{G},\mathscr{F})$$

If we take the direct image of $f^*\mathscr{G}$, we have the following equality

$$f_*f^*\mathscr{G}(W) = f^*\mathscr{G}(f^{-1}(W)) = \mathscr{G}(W)$$

Thus, we have the identity $f_*f^* = \mathrm{Id}_{\mathsf{Shv}(Y)}$. This implies that for any pre-sheaf morphism $\psi : f^*\mathscr{G} \to \mathscr{F}$, we can obtain a sheaf morphism

$$f_*(\psi) : \mathscr{G} = f_*f^*\mathscr{G} = B \to f_*\mathscr{F}$$

This defines the map β_2:

$$\beta_2 : \mathrm{Hom}_{\mathsf{PreShv}(X,R\text{-}\mathsf{Mod})}(f^*\mathscr{G},\mathscr{F}) \to \mathrm{Hom}_{\mathsf{Shv}(Y,R\text{-}\mathsf{Mod})}(\mathscr{G}, f_*\mathscr{F})$$

We claim that that β_1 and β_2 are inverses to each other. It is easy to see that $\beta_2 \circ \beta_1(\varphi) = \varphi$ by observing the following commutative diagram:

$$\begin{array}{ccc} \mathscr{G}(W) & \xrightarrow{\varphi(W)} & \mathscr{F}(f^{-1}(W)) \\ \big\downarrow & \nearrow_{(\beta_1(\varphi))(f^{-1}(W))} & \\ \varinjlim_{W \supseteq f(U)} \mathscr{G}(W) & & \end{array}$$

On the other hand, $\beta_1 \circ \beta_2(\psi) = \psi$ by the uniqueness from the universal property of the colimit. Diagram omitted. This completes the proof. $\square$

Part 3. Schemes

The theory of schemes is the foundation of modern algebraic geometry. It provides a unified framework that generalizes classical varieties and allows one to rigorously handle objects defined by arbitrary commutative rings.

14. Locally Ringed Spaces

Geometrically, a spectrum of a ring with its structure sheaf encodes both the topological space of prime ideals of R and the ring of functions defined locally on this space. This naturally leads to a more general framework: a category of geometric spaces equipped with a sheaf of rings. To capture this structure abstractly, we introduce the category of locally ringed spaces, which formalizes the essential features of affine schemes and provides the appropriate categorical setting in which schemes naturally reside. We first define a ringed space.

Definition 14.1. A **ringed space** is a pair $(X, \mathscr{F}_X)$, where X is a topological space and $\mathscr{F}_X$ is a CRing-valued sheaf on X. A morphism of ringed spaces from $(X, \mathscr{F}_X)$ to $(Y, \mathscr{G}_Y)$ is a pair $(f, f^{\#})$, where $f : X \to Y$ is a continuous map and

$$f^{\#} : \mathscr{G}_Y \to f_* \mathscr{F}_X$$

is a morphism of sheaves of rings on Y.

Remark 14.2. *The adjunction between the pushforward and pullback functors (Proposition 13.9) shows that the notion of a morphism of ringed spaces could equivalently have been defined in terms of the pullback of sheaves of commutative rings rather than pushforwards of sheaves of commutative rings. This perspective highlights the fundamental role of the adjunction.*

Remark 14.3. *The definition of morphisms of ringed spaces given in Definition 14.1 is inspired by the classical setting of affine varieties, where morphisms are continuous maps compatible with the pullback of regular functions. In particular, if X, Y are morphisms of affine varieties defined over an algebraically closed field, a morphism of affine varieties*

$$f : X \to Y$$

is precisely a continuous map such that the pullback defines a ring homomorphism

$$\mathscr{R}_Y(U) \to \mathscr{R}_X(f^{-1}(U)),$$

where $\mathscr{R}_{X,Y}$ are the sheaf of regular functions on X, Y respectively and $U \subseteq Y$ is an open set (Definition 4.7). In other words, we have a morphism of sheaves $\mathscr{R}_Y \to f_* \mathscr{R}_X$.

In a ringed space $(X, \mathscr{F}_X)$, it is natural to ask how to evaluate a section at a point. Given an open set $U \subseteq X$, a section $f \in \mathscr{F}_X(U)$, and a point $x \in U$, the germ $f_x \in \mathscr{F}_{X,x}$ captures the local behavior of f near x, but does not in itself define evaluation unless the stalks carry additional structure. In typical examples—such as the structure sheaf on $\operatorname{Spec} R$ or the sheaf of smooth functions on a manifold—the stalks are local rings. Each stalk $\mathscr{F}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x$ consisting of germs vanishing at x, and evaluation corresponds to the image of f_x in the residue field $\mathscr{F}_{X,x}/\mathfrak{m}_x$. This motivates the notion of a locally ringed space, where the stalks are required to be local rings, ensuring that evaluation at points is well-defined.

Definition 14.4. A ringed space $(X, \mathscr{F}_X)$ is a **locally ringed space** if for each point $x \in X$, the stalk $\mathscr{F}_{X,x}$ is a local ring with unique maximal ideal $\mathfrak{m}_x$. A morphism of locally ringed spaces $(f, f^\#) : (X, \mathscr{F}_X) \to (Y, \mathscr{F}_Y)$ is a morphism of ringed spaces such that the induced map

$$f_x^\# : \mathscr{F}_{Y,y} \to \mathscr{F}_{X,x}$$

maps $\mathfrak{m}_y$ into $\mathfrak{m}_x$ for every $x \in X$, $y \in Y$ such that $f(x) = y$.

Remark 14.5. *We now clarify the definition of a morphism between locally ringed spaces. Let $f : X \to Y$ be a morphism of topological spaces and suppose $x \in X$ with $f(x) = y$. The morphism of sheaves $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ induces, for every open subset $V \subseteq Y$, a ring homomorphism*

$$f^\#(V) : \mathscr{O}_Y(V) \to \mathscr{O}_X(f^{-1}(V)).$$

As V varies over all open neighborhoods of y, the preimages $f^{-1}(V)$ form a cofinal system of open neighborhoods of x. Passing to colimits, we obtain an induced map on stalks:

$$f_x^\# : \mathscr{O}_{Y,y} = \varinjlim_{y \in V} \mathscr{O}_Y(V) \longrightarrow \varinjlim_{x \in f^{-1}(V)} \mathscr{O}_X(f^{-1}(V)) = \mathscr{O}_{X,x}.$$

Hence, any morphism of ringed spaces $(f, f^\#)$ gives rise to a local map of stalks at each point $x \in X$. In the context of locally ringed spaces, we require that this induced map

$$f_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$$

be a local homomorphism of local rings[10].

Example 14.6. Let R be a ring. $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Remark 14.7. *It can be checked that locally ringed spaces assemble into a category, denoted* LocRing*. Let's verify that the composition of morphisms of locally ringed spaces is well-defined. Given morphisms*

$$(f, f^\#) : (X, \mathscr{F}_X) \to (Y, \mathscr{F}_Y),$$
$$(g, g^\#) : (Y, \mathscr{F}_Y) \to (Z, \mathscr{F}_Z).$$

we define the morphism $X \to Z$ by the composition $g \circ f$ on the level of topological spaces. The associated sheaf map $(g \circ f)^\#$ is defined, for an open set $U \subseteq Z$, by the composition

$$\mathscr{F}_Z(U) \xrightarrow{g_U^\#} \mathscr{F}_Y(g^{-1}(U)) \xrightarrow{f^\#_{g^{-1}(U)}} \mathscr{F}_X((g \circ f)^{-1}(U))$$

An isomorphism of locally ringed spaces is a morphism $f : X \to Y$. That is, f is a homeomorphism of topological spaces and for every open set $U \subseteq Y$, the map

$$f_U^\# : \mathscr{F}_Y(U) \to \mathscr{F}_X(f^{-1}(U))$$

is an isomorphism of rings.

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[10]That is, if $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ are local rings, a ring homomorphism $\varphi : R \to S$ is called local if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

15. Affine Schemes

Having introduced the spectrum of a ring and constructed the structure sheaf on this space, we are now ready to give the formal definition of an affine scheme.

Definition 15.1. Let R be a ring. An **affine scheme** is a pair $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ consisting of the topological space $\operatorname{Spec} R$ together with the structure sheaf, $\mathscr{O}_{\operatorname{Spec} R}$.

Example 15.2. Let $X = \operatorname{Spec} \mathbb{Z}$. Then the points of X are the prime ideals generated by prime numbers, together with the zero ideal. Since $\mathbb{Z}$ is a principal ideal domain (PID), every open subset of X is a distinguished open set. In particular, for the distinguished open set U_n, we have

$$\mathscr{O}_{\operatorname{Spec} \mathbb{Z}}(U_n) = \mathbb{Z}\left[\frac{1}{n}\right].$$

Note that X does not have an analogue in the category of (classical) affine varieties, AffVar.

Example 15.3. (Hartshorne II.2.11) We describe the affine scheme $\operatorname{Spec} \mathbb{F}_p[x]$. Since $\mathbb{F}_p[x]$ is a PID, the set of prime ideals is in 1-1 correspondence with irreducible monic polynomials in $\mathbb{F}_p[x]$. Therefore,

$$\operatorname{Spec} \mathbb{F}_p[x] = \{(0)\} \cup \{(f) \mid f \text{ is an irreducible monic polynomial in } \mathbb{F}_p[x]\},$$

When $f = 0$, then $\mathscr{O}_{\operatorname{Spec} \mathbb{F}_p[x],0} = \operatorname{Spec} \mathbb{F}_p[x]_{(0)} \cong \operatorname{Spec} \mathbb{F}_p(x)$ and the maximal idea $\mathfrak{m}_0$ is the zero ideal since $\operatorname{Spec} \mathbb{F}_p(x)$ is a field. When f is a non-zero irreducible monic polynomial of degree n, note that by definition,

$$\operatorname{Spec} \mathbb{F}_p[x]_{(f)} = \{g/h : g, h \in \operatorname{Spec} \mathbb{F}_p[x]\ f \nmid h\}$$

Recall that $\operatorname{Spec} \mathbb{F}_p[x]_{(f)}$ is a local ring since (f) is a prime ideal and that the unique maximal ideal of $\operatorname{Spec} \mathbb{F}_p[x]_{(f)}$ is given by:

$$\mathfrak{m}_{(f)} = \{a/b : a, b \in \operatorname{Spec} \mathbb{F}_p[x]\ f \nmid b\ f \mid a\}$$

Here $\mathfrak{m}_{(f)}$ is the ideal $\mathfrak{m} = (f)$ localized at (f).

Let AffSch be the full subcategory of $\mathsf{LocRing}$ consisting of locally ringed spaces isomorphic to the spectrum of some ring. We now come to the all-important result: the category AffSch is equivalent to CRing opposite category of commutative rings, establishing a deep duality between algebra and geometry.

Proposition 15.4. *AffSch is equivalent to the category $\mathsf{Rings}^{\mathrm{op}}$.*

Remark 15.5. *We write $\mathsf{AffSch} \simeq \mathsf{CRing}^{\mathrm{op}}$.*

Proof. Consider the functor:

$$\Gamma : \mathsf{AffSch} \longrightarrow \mathsf{Rings}^{\mathrm{op}}$$
$$(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R}) \mapsto \mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) = R$$

We show that Γ is fully faithful and essentially surjective. The latter follows since Proposition 11.8 implies that $\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$. We now show that Γ is fully faithful by showing that for any $R, S \in \mathsf{CRing}$, the map

$$\tau_{R,S} : \operatorname{Hom}_{\mathsf{Rings}}(R, S) \longrightarrow \operatorname{Hom}_{\mathsf{AffSch}}(\operatorname{Spec} S, \operatorname{Spec} R)$$

defined by Lemma 5.12 is bijective. Let $\varphi : R \to S$ be a ring homomorphism, and let $f : \operatorname{Spec} S \to \operatorname{Spec} R$ induced map as in Lemma 5.12. We must show that f induces a morphism on the structure sheaves in order for it to define a morphism of locally ringed spaces. It suffices to define the morphism

$$f^{\#} : \mathscr{O}_{\operatorname{Spec} R} \to f_*(\mathscr{O}_{\operatorname{Spec} S})$$

on distinguished open sets. First note that

$$\mathscr{O}_{\operatorname{Spec} R}(U_h) \cong R_h,$$
$$\mathscr{O}_{\operatorname{Spec} R}(f^{-1}(U_h)) \cong S_{\varphi(h)}.$$

There is a ring homomorphism $R \to S \to S_{\varphi(g)}$. Since the image of g is invertible in $S_{\varphi(g)}$, the universal property of localization gives an induced ring homomorphism $R_g \to S_{\varphi(g)}$. This defines $f^{\#}$ and shows that $\tau_{R,S}$ is well-defined. The candidate for the inverse map is given by

$$\gamma_{R,S} : \operatorname{Hom}_{\mathsf{AffSch}}(\operatorname{Spec} S, \operatorname{Spec} R) \longrightarrow \operatorname{Hom}_{\mathsf{Rings}}(R, S)$$

defined by taking global sections. It is clear that $\gamma_{R,S} \circ \tau_{R,S}$ is the identity map. We now show that $\tau_{R,S} \circ \gamma_{R,S}$ is the identity map. Suppose given a morphism of locally ringed spaces

$$(f, f^{\#}) : \operatorname{Spec} S \to \operatorname{Spec} R$$

Taking global sections, $f^{\#}$ induces a homomorphism of rings $\varphi : R \to S$. Given $\mathfrak{p} \in \operatorname{Spec} S$, we obtain a morphism of local rings on stalks, which is compatible with φ and localization, yielding the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ R_{f(\mathfrak{p})} & \longrightarrow & S_{\mathfrak{p}} \end{array}$$

Since $f^{\#}$ is a local homomorphism, it follows that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$ [11]. This shows that the underlying map f coincides with the canonical map

$$\operatorname{Spec} R \xrightarrow{f} \operatorname{Spec} S$$

induced by φ. It then follows that $f^{\#}$ is also the structure sheaf morphism induced by φ, so that the morphism $(f, f^{\#})$ of locally ringed spaces is indeed induced by the ring homomorphism φ. This shows $\tau_{R,S} \circ \gamma_{R,S}$ is the identity map □

Example 15.6 (Hartshorne II.2.5)**.** $\mathbb{Z}$ is the initial object in CRing because any commutative ring R, there exists a unique ring homomorphism

$$\varphi : \mathbb{Z} \to R$$
$$1 \mapsto 1_R$$

Therefore, Proposition 15.4 implies that $\operatorname{Spec} \mathbb{Z}$ is the final object in AffSch. It follows that in AffSch, for any affine scheme $\operatorname{Spec} R$, there exists a unique morphism

$$\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}.$$

[11] Suppose $r \notin f(\mathfrak{p})$. Then the image of r in $R_{f(\mathfrak{p})}$ is a unit, so $f_p^{\#}(r)$ is a unit in $S_{\mathfrak{p}}$. Hence, $\varphi(r) \notin \mathfrak{p}$, i.e., $r \notin \varphi^{-1}(\mathfrak{p})$. Conversely, assume $r \notin \varphi^{-1}(\mathfrak{p})$, so that $\varphi(r) \notin \mathfrak{p}$, and thus the image of $\varphi(r)$ is a unit in $S_{\mathfrak{p}}$. Since $f_p^{\#} : R_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$ is a local homomorphism, the preimage of a unit is a unit. Therefore, the image of r in $R_{f(\mathfrak{p})}$ is a unit, which implies $r \notin f(p)$. Thus, we conclude that $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$.

We have established the category of affine schemes and demonstrated the fundamental result:

The category of affine schemes is equivalent
to the opposite category of commutative rings.

However, this does not capture the full scope of schemes. To illustrate the limitation, consider the analogy with complex manifolds, which are geometric spaces locally modeled on $(\mathbb{C}^n, \mathscr{H})$, where $\mathscr{H}$ denotes the sheaf of holomorphic functions. In this analogy, the category of affine schemes corresponds to the local models $(\mathbb{C}^n, \mathscr{H})$. Therefore, a *general scheme* is a topological space equipped with a sheaf of rings that is locally isomorphic to an affine scheme. Schemes are discussed in the next section.

16. General Schemes

A scheme is a locally ringed space that is locally isomorphic to an affine scheme. This construction mirrors the classical notion of a smooth manifold: whereas a manifold is a topological space locally modeled on open subsets of $\mathbb{R}^n$, equipped with a sheaf of smooth functions, a scheme is built by gluing together spectra of rings, each with a corresponding sheaf of regular functions. This local-to-global approach allows schemes to capture both geometric and aripropetic information in a unified framework.

Definition 16.1. A **scheme** is a locally ringed space $(X, \mathscr{F}_X)$ such that every $x \in X$ has an open neighborhood $x \in U_x$ such that $(U_x, \mathscr{F}_X|_{U_x})$ is an affine scheme.

Example 16.2. Let R be a ring. Then $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is a (affine) scheme.

A morphism of schemes is defined as a morphism of the underlying locally ringed spaces. That is, it consists of a continuous map between the underlying topological spaces together with a morphism of structure sheaves that respects the local ring structure at each point. This definition places the category of schemes, denoted by Sch, as a subcategory of the category of locally ringed spaces, $\mathsf{LocRing}$.

Remark 16.3. *The category of affine schemes, denoted* AffSch*, is a full subcategory of* Sch*.*

Remark 16.4. *If* $f : (X, \mathscr{F}_X) \to (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ *is a morphism of schemes, there is an induced morphism of sheaves*

$$f^\# : \mathscr{O}_{\operatorname{Spec} R} \to f_* \mathscr{F}_X,$$

which on global sections induces a ring homomorphism

$$R \cong \Gamma(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R}) \to \Gamma(X, \mathscr{F}_X).$$

This yields a map

$$\alpha : \operatorname{Hom}((X, \mathscr{F}_X), (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})) \to \operatorname{Hom}(R, \Gamma((X, \mathscr{F}_X))).$$

A generalization of Proposition 15.4 states that the map α *is bijective. In particular, this shows that* $\operatorname{Spec} \mathbb{Z}$ *is a final object in the category* Sch *since* $\mathbb{Z}$ *is initial in* CRing*. Similarly,* $\operatorname{Spec} 0$ *is initial in* Sch *since* 0 *is final in* CRing*.*

16.1. **Basic Topological Properties.** We establish some basic topological properties of schemes. The topological space underlying a scheme is not generally expected to be Hausdorff. Indeed, as we have seen, even an affine scheme is not Hausdorff. However, an affine scheme is a T_0-space, and since any scheme can be covered by affine open subsets, we expect that a general scheme is also a T_0-space.

Proposition 16.5. *Let X be a scheme. Then X is a T_0-space*

Proof. e can cover X by affine open subsets. If x and y are contained in a common affine open subset, then the T_0-property follows from the fact that any affine scheme is a T_0-space (Proposition 6.4). Otherwise, there exists an affine open neighborhood of one point, say x, which does not contain the other point y. □

We now turn to a basic property of generic points in a scheme.

Proposition 16.6. *[Hartshorne II.2.9] If X is a scheme, then every non-empty irreducible closed subset $Z \subseteq X$ has a unique generic point, ξ.*

Proof. The proof is given below:

(1) Assume first that X is an affine scheme. Since $Z \subset X$ is a closed and irreducible subset, there exists a prime ideal $\mathfrak{p} \subset A$ such that $Z = V(\mathfrak{p})$. In particular, $\mathfrak{p} \in Z$, and by Proposition 6.1, we have
$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = Z.$$
(2) Now assume that X is a general scheme. Let $x \in Z \neq \emptyset$, and let $U \subset X$ be a non-empty affine open subset containing x. Consider the set $V := U \cap Z$. Then V is a non-empty closed and irreducible subset of U. By (1), there exists a point $\xi \in U$ such that
$$\overline{\{\xi\}} = V \subseteq Z,$$
where the closure is taken in U. Consider $\overline{\{\xi\}}$ in X. Observe that we can write Z as the union of two closed subsets:
$$Z = \overline{\{\xi\}} \cup (U^c \cap Z)\,.$$
Since Z is irreducible and $Z \neq U^c \cap Z$, it follows that $Z = \overline{\{\xi\}}$.
(3) We now show uniqueness. Assume that there exist two distinct points $\xi_{1,2}$ such that $\overline{\{\xi_{1,2}\}} = Z$. Since X is a T_0-space (Proposition 16.5), there exists an open subset $U \subseteq X$ such that $\xi_1 \in U$ and $\xi_2 \notin U$. However, this contradicts the assumption that $\xi_1 \in \overline{\{\xi_2\}}$, which implies that every open neighborhood of ξ_1 must also contain ξ_2. Thus, $\xi_1 = \xi_2$.

This completes the proof. □

16.2. **Relative Schemes.** Let S be a scheme. A scheme over S is simply a scheme X equipped with a morphism
$$X \to S$$
A scheme over S is also called as S-scheme. S is also called the base scheme. If we have an S-scheme, then the morphism $X \to S$ allow us to think of X as being defined relative to S,

much like families of algebraic varieties or solutions to polynomial equations depending on parameters. A morphism of schemes over S is then a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

This allows us to define the category of schemes over a base scheme S, denoted by Sch_S. Note that Sch_S is the slice category of Sch over S.

Example 16.7. let $\mathbb{K}$ be a field and let V be an affine variety, that is, the spectrum of a finitely generated $\mathbb{K}$-algebra

$$A(V) = \mathbb{K}[x_1, \dots, x_n]/\mathfrak{p},$$

where $\mathfrak{p}$ is a prime ideal. The inclusion $\mathbb{K} \hookrightarrow A(V)$ induces a morphism of schemes

$$V \to \operatorname{Spec} \mathbb{K}.$$

17. Open and Closed Subschemes

We discuss open and closed subschemes. These constructions are essential for understanding how schemes are built from and relate to their local pieces.

17.1. **Open Subschemes.**

Recall that an open subset of a smooth manifold naturally inherits the structure of a manifold. We ask whether an analogous statement holds for schemes.

Proposition 17.1. *[Hartshorne II.2.1 & II.2.2] Let $(X, \mathscr{F}_X)$ be a scheme.*

(1) *If $(X, \mathscr{F}_X) \cong (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is an affine scheme, then the locally ringed space $(U_f, \mathscr{O}_X|_{U_f})$ is isomorphic to $(\operatorname{Spec} R_f, \mathscr{O}_{\operatorname{Spec} R_f})$.*
(2) *Let $U \subseteq X$ be any open subset. Then $(U, \mathscr{F}_X|_U)$ is a scheme.*

Proof. The proof is given below:

(1) Recall that we have the following bijective correspondence:

$$\{\text{Prime ideals of } R_f\} \longleftrightarrow \{\text{Prime ideals of } R \text{ that don't contain } f\}$$

Hence, $\operatorname{Spec} R_f \cong U_f$ as sets. This correspondence extends to the level of topological spaces. Indeed, $\{\operatorname{Spec} R_g\}_{g \in R}$ is a basis for the topology on $\operatorname{Spec} R$. On the other hand, the sets

$$\{\operatorname{Spec} R_f \cap \operatorname{Spec} R_g\}_{g \in R}$$

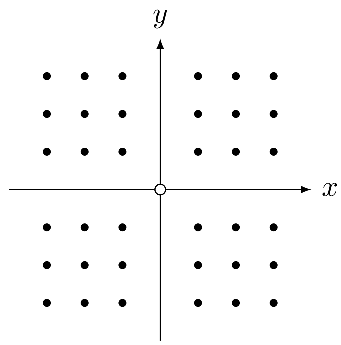
form a basis for the topology on $\operatorname{Spec} R_f$, and this is also a basis for the subspace topology on U_f. To conclude that $\operatorname{Spec} R_f \cong U_f$ as locally ringed spaces, it remains to verify that the structure sheaves agree under this identification. This follows from the construction of the structure sheaf: the stalks at a prime ideal $\mathfrak{p}$ in U_f are isomorphic to $R_{\mathfrak{p}}$, and since $f \notin \mathfrak{p}$, the element f is invertible in $R_{\mathfrak{p}}$. Thus, localization at f does not affect the local behavior at such prime ideals.

(2) For each $x \in U$ let $x \in V_x$ be an open set such that

$$(V_x, \mathscr{F}_X|_{V_x}) \cong (\operatorname{Spec} R_x, \mathscr{O}_{\operatorname{Spec} R_x}).$$

Because $x \in U \cap V_x$ is an open set in $(\operatorname{Spec} R_x, \mathscr{O}_{\operatorname{Spec} R_x})$, there exists a distinguished open set $x \in U_f \subseteq U \cap V_x$. By (1),

$$(U_f, \mathscr{O}_X|_{U_f}) \cong (\operatorname{Spec} R_f, \mathscr{O}_{\operatorname{Spec} R_f}).$$



The scheme $\operatorname{Spec} \mathbb{K}[x,y] \setminus \{(0,0)\}$

Thus $(U, \mathscr{F}_X|_U)$ is a scheme.

This completes the proof. □

Remark 17.2. *Proposition 17.1 implies that the underlying topological space of every scheme has a base of open affine schemes*

Remark 17.3. *We refer to $(U, \mathscr{O}_X|_U)$ as an open sub-scheme of X.*

Example 17.4. The open set $U = \mathbb{A}^1 \setminus V(x)$ is an open subscheme of the affine line $\mathbb{A}^1_k = \operatorname{Spec}(K[x])$. Note the isomorphism of schemes (Example 3.14)

$$U \cong \operatorname{Spec}(\mathbb{K}[x, x^{-1}]) = \operatorname{Spec}\left(\frac{\mathbb{K}[x,y]}{(xy-1)}\right).$$

Armed with the notion of an open subscheme, we are now ready to examine an example of a scheme that is not affine. This example is geometrically well-motivated and illustrates that the passage from affine schemes to general schemes is both natural and necessary.

Example 17.5. Let $\mathbb{K}$ be a field. Consider the affine scheme $(\operatorname{Spec} \mathbb{K}[x,y], \mathscr{O}_{\operatorname{Spec} \mathbb{K}[x,y]})$. Let

$$X = \operatorname{Spec} \mathbb{K}[x,y] - \{(x,y)\}$$

Note that $X = U_x \cup U_y$. Hence, X is an open set of our affine scheme. Therefore, $(X, \mathscr{O}_{\operatorname{Spec} \mathbb{K}[x,y]}|_X)$ is an open sub-scheme. We show that this sub-scheme is not an affine scheme. We compute $\mathscr{O}_{\operatorname{Spec} \mathbb{K}[x,y]}|(X)$. We find rational functions defined on U_x and U_y that agree on the intersection $U_x \cap U_y = U_{xy}$. Clearly, rational functions that have only powers of x in the denominator and also only powers of y in the denominator must, in fact, be polynomials. Thus, we conclude[12]:

$$\mathscr{O}_{\operatorname{Spec} \mathbb{K}[x,y]}(X) \cong \mathbb{K}[x,y].$$

More precisely, we have the following sequence of rings:

$$0 \longrightarrow \mathscr{O}_{\operatorname{Spec} \mathbb{K}[x,y]}(X) \xrightarrow{\alpha} \mathbb{K}[x,y]_x \oplus \mathbb{K}[x,y]_y \xrightarrow{\beta} \mathbb{K}[x,y]_{xy}$$

Here $\alpha(s) = \left(s|_{U_x},\ s|_{U_y}\right)$ and $\beta(a/x^n, b/y^m) = a/x^n - b/y^m$. $\mathscr{O}_{\operatorname{Spec} \mathbb{K}[x,y]}(X)$ is identified with the kernel of β, which consists of pairs $(a/x^n, b/y^m)$ such that

$$\frac{a}{x^n} = \frac{b}{y^m} \quad \text{in } \mathbb{K}[x,y]_{xy}.$$

[12]In other words, the removal of the origin does not introduce any new global regular functions.

This equality implies that $ay^m = bx^n$ in $\mathbb{K}[x, y]$. Since $\mathbb{K}[x, y]$ is a UFD, we conclude that

$$a = cx^n \quad \text{and} \quad b = cy^m$$

for some $c \in \mathbb{K}[x, y]$. Hence, any element in the kernel is of the form (c, c) with $c \in \mathbb{K}[x, y]$. Hence, $\mathscr{O}_{\operatorname{Spec}\mathbb{K}[x,y]}(X) \cong \mathbb{K}[x, y]$. If X were affine, then

$$\operatorname{Spec}\mathbb{K}[x, y] - \{(x, y)\} \cong X \cong \mathscr{O}_X(X) \cong \operatorname{Spec}\mathbb{K}[x, y]$$

However, X is not homeomorphic to $\operatorname{Spec}\mathbb{K}[x, y]$. Every proper ideal in an affine scheme has a nonempty vanishing locus, yet the ideal (x, y) has empty vanishing locus in X.

We now study an important class of morphisms of schemes, namely, open immersions. Open immersions correspond to the inclusion of open subschemes, and as such, reflect the local nature of the scheme structure.

Definition 17.6. Let $(X, \mathscr{F}_X), (Y, \mathscr{F}_Y)$ be schemes. An **open immersion** is a morphism of schemes $f : (X, \mathscr{F}_X) \to (Y, \mathscr{F}_Y)$ such that f induces an isomorphism of locally ringed spaces $\rho : (X, \mathscr{F}_X) \xrightarrow{\sim} (U, \mathscr{F}_Y|_U)$, where $U \subseteq Y$ is an open subset. That is, f factors as the composition

$$\begin{array}{ccc} (X, \mathscr{F}_X) & \xrightarrow[\sim]{\rho} & (U, \mathscr{F}_Y|_U) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & (Y, \mathscr{F}_Y) & \end{array}$$

Remark 17.7. *Note that if X is a subset of Y, then the image under the open immersion, which can be identified with $(X, \mathscr{F}_X)$, is an open subscheme of $(Y, \mathscr{F}_Y)$. The difference between open immersions and open subschemes is a bit confusing, and not too important: at the level of sets, open subschemes are subsets, while open immersions are bijections onto subsets.*

Proposition 17.8. *The following is a basic list of properties of open immersions.*

(1) A composition of open immersions is an open immersion.
(2) If $f : X \to Y$ is an open immersion, then for any open subset V of Y, the restricted morphism $f^{-1}(V) \to V$ is an open immersion.
(3) If $f : X \to Y$ is a morphism of schemes such that if there is an open cover $\{V_i\}$ of Y for which each restricted morphism $f^{-1}(V_i) \to V_i$ is an open immersion, then f is in an open immersion.

Proof. (1) is clear since an open subscheme of an open subscheme is an open subscheme (of the original scheme). (2) follows using the fact that a non-empty intersection of open subschemes $(U, \mathscr{F}_Y|_U)$ and $(V, \mathscr{F}_Y|_V)$ is an open subscheme $(U \cap V, \mathscr{F}_Y|_{U\cap V})$. (3) follows because an arbitrary union of open subschemes is an open subscheme, and that isomorphisms of schemes follow the propety given by assumption. □

Remark 17.9. *$f : X \to Y$ is an open immersion. If Y is compact, then X need not be compact. Indeed, consider*

$$f : \bigcup_{n\geq 1} U_{x_n} \hookrightarrow \operatorname{Spec}(\mathbb{K}[x_1, x_2, \dots])$$

Clearly, f is an open immersion since it is an isomorphism onto the open subscheme determined by $\bigcup_{n\geq 1} D(x_n)$. $\operatorname{Spec}(\mathbb{K}[x_1, x_2, \dots])$ is compact, but $\bigcup_{n\geq 1} D(x_n)$ is not compact.

Additionally, if $f : X \to Y$ is a morphism of schemes such that if there is an open cover $\{V_i\}$ of X for which each restricted morphism $V_i \to f(V_i)$ is an open immersion, then f need not be in an open immersion. Consider:

$$f : \operatorname{Spec} \mathbb{K}[x] \coprod \operatorname{Spec} \mathbb{K}[y] \hookrightarrow \operatorname{Spec} \mathbb{K}[z]$$

be defined by the obvious inclusion map on each component [13]*. Clearly, f is an open immersion when restricted to* $\operatorname{Spec} \mathbb{K}[x]$ *or* $\operatorname{Spec} \mathbb{K}[y]$*. However, f is not an open immersion itself since it is not even an injective map of the underlying sets.*

17.2. **Closed Subschemes.** Intuitively, we expect a closed subscheme of a scheme X to be a scheme Z together with a morphism $Z \hookrightarrow X$ that identifies Z with a closed subset of X, endowed with a scheme structure compatible with that of X. However, a given closed subset may admit multiple distinct scheme structures. This fact introduces additional subtlety in the definition of a closed subscheme, in contrast to the relatively straightforward case of open subschemes. The prototypical example we aim to formalize as a closed subscheme is given by closed subsets of $\operatorname{Spec} R$ for some ring R. Let R be a ring and let $\mathfrak{a}$ be an ideal of R. If $\mathfrak{a}$ is an ideal of R, then the ring homomorphism $R \to R/\mathfrak{a}$ induces a morphism of schemes

$$f : \operatorname{Spec} R/\mathfrak{a} \to \operatorname{Spec} R$$

Since the $R \to R/\mathfrak{a}$ is surjective, the map f is a homeomorphism onto the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec} R$, and the map of structure sheaves

$$\mathscr{O}_{\operatorname{Spec} R} \to f_* \mathscr{O}_{\operatorname{Spec} R/\mathfrak{a}}$$

is surjective as we now show.

Lemma 17.10. *[HartshornII.2.18(c)] Let $\varphi : R \to S$ be a surjective ring homomorphism. If $Y = \operatorname{Spec} S$ and $X = \operatorname{Spec} R$, then $f : Y \to X$ is a homeomorphism of Y onto a closed subset and the morphism*

$$f^{\#} : \mathscr{O}_X \to f_* \mathscr{O}_Y$$

is surjective.

Proof. We have $S \cong R/\ker \varphi$ which implies that

$$\operatorname{Spec} S \cong \operatorname{Spec}(R/\ker \varphi) = V(\ker \varphi),$$

Since we already know that the induced map f is continuous and its image is the closed subset $V(\ker \varphi)$, it follows that f is a homeomorphism onto its image, which is a closed subset of $\operatorname{Spec} R$. It suffices to show that $f^{\#}$ is surjective on stalks[14]. Let $\mathfrak{q} \in \operatorname{Spec} S$, and consider the corresponding prime ideal $\varphi^{-1}(\mathfrak{q}) \in \operatorname{Spec} R$. Then we have the identifications

$$\mathscr{O}_{\operatorname{Spec} R, \varphi^{-1}(\mathfrak{q})} \cong R_{\varphi^{-1}(\mathfrak{q})},$$
$$\mathscr{O}_{\operatorname{Spec} S, \mathfrak{q}} \cong S_{\mathfrak{q}}.$$

Since $\varphi : R \to S$ is surjective, the induced map on localizations $R_{\varphi^{-1}(\mathfrak{q})} \to S_{\mathfrak{q}}$ is also surjective. It follows that the morphism of sheaves $f^{\#}$ is surjective. □

[13]Disjoint union of schemes is a coproduct in the category of schemes so f is indeed well-defined.
[14]This statement is proved in other notes, which discuss further details on sheaves.

Remark 17.11. *Let $Y = \operatorname{Spec} S$ and $X = \operatorname{Spec} R$. We also have a converse to Lemma 17.10. That is, if $f : Y \to X$ is a morphism that is a homeomorphism onto a closed subset of X, and the induced morphism of sheaves*

$$f^{\#} : \mathscr{O}_X \to f_* \mathscr{O}_Y$$

is surjective, then f is a closed immersion. This is [Har13, Exercise II.2.18(d)].

The discussion above motivates the following definition:

Definition 17.12. Let $(X, \mathscr{F}_X)$ and $(Z, \mathscr{F}_Z)$ be schemes. A morphism $\iota : Z \to X$ is called a **closed immersion** if there exists an open cover $\{U_i\}_{i\in I}$ of subschemes of X such that for each $i \in I$:

(1) the preimage $\iota^{-1}(U_i)$ is affine subscheme, and
(2) the induced ring homomorphism
$$\iota^{\#} : \mathscr{F}_X(U_i) \to \mathscr{F}_Z(\iota^{-1}(U_i))$$
is surjective.

We say that Z is a **closed subscheme** of X.

Concretely, the schemes X and Z can be covered by affine open subsets $U_i = \operatorname{Spec}(R_i)$, such that $\iota^{-1}(U_i) = \operatorname{Spec}(S_i)$ for each i. The induced ring homomorphism $R_i \to S_i$ is surjective, which implies that $S_i \cong R_i/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i \subseteq R_i$. Consequently, the restriction $\iota^{-1}(U_i) \to U_i$ corresponds to the morphism of affine schemes

$$\operatorname{Spec}(R_i/\mathfrak{a}_i) \to \operatorname{Spec} R_i.$$

Remark 17.13. *A composition of closed immersions is a closed immersion. This is clear. Moreover, if $f : X \to Y$ is a morphism of schemes such that if there is an open cover $\{V_i\}$ of Y for which each restricted morphism $f^{-1}(V_i) \to V_i$ is a closed immersion, then f is in a closed immersion. (1) in Definition 17.12 can be checked easily. On the induced morphism of sheaves, we have:*

$$(f_i)_*(\mathscr{F}_Z\big|_{f^{-1}(U_i)}) = (f_*\mathscr{F}_Z)\big|_{U_i}$$

Hence f_* is surjective if and only if $(f_i)_*$ is surjective for all i.

18. Gluing Schemes

We will encounter a variety of constructions that demonstrate how to glue schemes and morphisms of schemes. These are essential techniques that enable us to build schemes and morphisms from local data, following a bottom-up approach. We begin by discussing how to glue morphisms of schemes when given morphisms defined on an open cover of the underlying topological space of a scheme.

Proposition 18.1. *Let X and Y be schemes, and let $\mathscr{B}$ be a basis for the topology on X. Suppose we are given a family of morphisms*

$$\{f_U : U \to Y\}_{U\in\mathscr{B}},$$

such that for all $V, U \in \mathscr{B}$ with $V \subseteq U$, the restriction satisfies

$$f_U|_V = f_V.$$

Then there exists a unique morphism of schemes $\varphi : X \to Y$ such that for every $U \in \mathscr{B}$, the restriction of φ to U agrees with φ_U, i.e.,

$$f|_U = f_U.$$

Proof. Define a map $f : X \to Y$ by setting $f(x) := f_U(x)$, where $U \in \mathscr{B}$ is any open neighborhood of x. The assumption $f_U|_V = f_V$ for $V \subseteq U$ ensures this is well-defined. Since each f_U is continuous and $\mathscr{B}$ is a basis, it follows that f is continuous. We now define a morphism of sheaves

$$f^{\#} : \mathscr{F}_Y \to f_* \mathscr{F}_X.$$

Let $W \subseteq Y$ be an open set. For any basic open set $U \in \mathscr{B}$ such that $U \subseteq f^{-1}(W)$, we have a morphism

$$f_U^{\#}(W) : \mathscr{F}_Y(W) \longrightarrow \mathscr{F}_X|_U\big(f_U^{-1}(W)\big) = \mathscr{F}_X\big(f^{-1}(W) \cap U\big) = \mathscr{F}_X(U).$$

If $V \subseteq U$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{F}_Y(W) & \xrightarrow{f_U^{\#}(W)} & \mathscr{F}_X(U) \\ & \searrow^{f_V^{\#}(W)} & \downarrow \mathrm{res}_{U,V} \\ & & \mathscr{F}_X(V) \end{array}$$

which is induced on sheaves by the relation

$$f_V = f_U|_V = f_U \circ i_V,$$

where $i_V : V \hookrightarrow U$ is the inclusion. For any element $s \in \mathscr{F}_Y(W)$, the family of sections

$$\big\{ f_U^{\#}(W)(s) \in \mathscr{F}_X(U) \big\}_{U \in \mathscr{B},\, U \subseteq f^{-1}(W)}$$

defines a unique element of $\mathscr{F}_X(f^{-1}(W))$. It is clear that this definition is in fact a homomorphism. □

Corollary 18.2. *Let X and Y be schemes, and let $\mathscr{B}$ be a base for the topology on X. Suppose we have a family of open subsets $\{U_W\}_{W \in \mathscr{B}}$ covering Y such that*

$$U_V \subseteq U_W \quad \textit{whenever } V \subseteq W \textit{ in } \mathscr{B}.$$

Assume there is a family of morphisms of schemes

$$f_W : W \to U_W, \quad \textit{for each } W \in \mathscr{B},$$

satisfying the following compatibility conditions if $V \in \mathscr{B}$ is contained in WL

1. $f_W^{-1}(U_V) = V$,
2. $f_W|^{U_V} = f_V$.

Then there exists a unique morphism of schemes $f : X \to Y$ such that

$$f|_W = f_W \quad \textit{for all } W \in \mathscr{B}.$$

Proof. For any $W \in \mathscr{B}$, let $i_W : U_W \hookrightarrow Y$ denote the inclusion morphism. The family of morphisms

$$g_W := i_W \circ f_W : W \to Y$$

satisfies the hypotheses of Proposition 18.1. □

Remark 18.3. *A similar argument as in Proposition 18.1 shows that if X and Y are schemes, and $\{U_i\}_{i \in I}$ is an open cover of X by open subschemes, then a compatible family of morphisms $f_i : U_i \to Y$ can be uniquely glued to obtain a morphism of schemes $f : X \to Y$.*

We now turn to the central construction: the gluing of schemes. The overarching idea is as follows. Suppose we are given a family of schemes $\{X_i\}_{i\in I}$, possibly infinite, which we intend to assemble into a single scheme X. To understand this construction on the level of topological spaces, observe that to incorporate each X_i into X as a subspace, a natural starting point is the disjoint union $\bigsqcup_i X_i$. This construction includes each X_i as an open subset, but as a disjoint union, the images of different X_i remain separate. In general, however, we desire the images of distinct schemes X_i and X_j to intersect in a meaningful way within X. Therefore, to achieve a genuine gluing, it is necessary to impose additional data describing the local identifications. Specifically, for each pair (i, j), we select open subschemes $U_{ij} \subseteq X_i$ that play the role of the intersection $X_i \cap X_j$ inside X. These open subsets will serve to identify parts of X_i and X_j compatibly, providing the local structure needed to glue the family $\{X_i\}$ into a scheme X.

Proposition 18.4. (Hartshorne II.2.12) *Let $\{X_i\}_{i\in I}$ be a possibly infinite. family of scheme. For each pair of indices $i, j \in I$, suppose we are given an open subscheme $U_{ij} \subseteq X_i$. Assume further that for each $i, j \in I$, we are given an isomorphism of schemes*

$$\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$$

satisfying the following conditions:

1. *For each i, $\varphi_{ii} = \mathrm{Id}_{X_{ii}}$.*
2. *For each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$;*
3. *For each i, j, and k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$,*
4. *$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$*

There is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that:

1. *ψ_i is an isomorphism of X_i onto an open subscheme of X.*
2. *The $\psi_i(X_i)$'s cover X.*
3. *$\psi_i(U_{ij}) = \psi_i(U_i) \cap \psi_j(X_j)$.*
4. *$\psi_i|_{U_{ij}} = \psi_j|_{U_{ji}} \circ \varphi_{ij}$*

Proof. We begin by constructing a topological space, which we then equip with a sheaf of commutative rings. Let Y denote the disjoint union of the topological spaces underlying the schemes X_i. Define an equivalence relation on Y by declaring:

$$x \sim y \iff \varphi_{ij}(x) = y$$

whenever $x \in U_{ij} \subseteq X_i$ and $y \in U_{ji} \subseteq X_j$, with $\varphi_{ij} : U_{ij} \to U_{ji}$ an isomorphism of schemes. Let $X = Y/\sim$ be the quotient topological space, and let $\pi : Y \to X$ be the natural projection. A subset $U \subseteq X$ is declared open if and only if $\pi^{-1}(U)$ is open in Y. Since Y is the disjoint union of the X_i, this is equivalent to the condition that $\pi^{-1}(U) \cap X_i$ is open in X_i for all i. In particular, for each i, the set $\pi(X_i) \subseteq X$ is open, because

$$\pi^{-1}(\pi(X_i)) \cap X_j = U_{ji}$$

for all j. It follows that the composition of the inclusion $X_i \hookrightarrow Y$ with the projection π defines a continuous map $\psi_i : X_i \to X$, which is a homeomorphism onto its image. Denote the open subset $\psi_i(X_i) \subseteq X$ by W_i. Then the sets $\{W_i\}$ form an open cover of X. Moreover, the compatibility condition $\psi_i = \psi_j \circ \varphi_{ij}$ holds on U_{ij}, and we observe that

$$\psi_i(U_{ij}) = W_i \cap W_j.$$

Now define a sheaf of commutative rings $\mathscr{F}_i$ on W_i by pushforward along ψ_i:

$$\mathscr{F}_i = \psi_{i,*}\mathscr{F}_{X_i},$$

that is, for any open subset $U \subseteq W_i$, we set

$$\mathscr{F}_i(U) = \mathscr{F}_{X_i}(\psi_i^{-1}(U)).$$

For any open set $V \subseteq X_j$ contained in U_{ji}, the isomorphism of schemes φ_{ij} induces an isomorphism of structure sheaves:

$$\varphi_{ji}^{\#V} : \mathscr{F}_{X_j}(V) \to \mathscr{F}_{X_i}(\varphi_{ij}^{-1}(V)).$$

Observe that for any open set $U \subseteq W_i \cap W_j$, we have $\psi_j^{-1}(U) \subseteq U_{ji}$, and moreover,

$$\varphi_{ij}^{-1}(\psi_j^{-1}(U)) = (\psi_j \circ \varphi_{ij})^{-1}(U) = \psi_i^{-1}(U).$$

Hence, for such U, the map $\varphi_{ji}^{\#}$ yields an isomorphism

$$\varphi_{ji}^{\#,\psi_j^{-1}(U)} : \mathscr{F}_{X_j}(\psi_j^{-1}(U)) \to \mathscr{F}_{X_i}(\psi_i^{-1}(U)),$$

which induces an isomorphism of sheaves

$$\sigma_{ji} : \mathscr{F}_j|_{W_i \cap W_j} \to \mathscr{F}_i|_{W_i \cap W_j}.$$

The collection $\{\sigma_{ij}\}$ satisfies the cocycle conditions: for each i, we have $\sigma_{ii} = \mathrm{id}$, since $\varphi_{ii}^{\#}$ is the identity. Moreover, for any triple i, j, k, the following cocycle condition holds on $W_i \cap W_j \cap W_k$:

$$\sigma_{ik} = \sigma_{jk} \circ \sigma_{ij}.$$

That is, for any open subset $U \subseteq W_i \cap W_j \cap W_k$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}_i(U) & \xrightarrow{\sigma_{ij}} & \mathscr{F}_j(U) \\ & \searrow^{\sigma_{ik}} & \downarrow{\scriptstyle \sigma_{jk}} \\ & & \mathscr{F}_k(U) \end{array}$$

Therefore, the family of sheaves $\{\mathscr{F}_i\}$ satisfies the gluing conditions. By the sheaf-gluing theorem, there exists a unique sheaf of rings $\mathscr{F}_X$ on X, together with isomorphisms

$$\psi_i^{\#} : \mathscr{F}_X|_{W_i} \xrightarrow{\sim} \mathscr{F}_i$$

such that on each overlap $W_i \cap W_j$, the transition condition

$$\psi_j^{\#} = \sigma_{ij} \circ \psi_i^{\#}$$

is satisfied. Thus, $(X, \mathscr{F}_X)$ is a locally ringed space. Since each $\psi_i : X_i \to X$ is an open immersion, and since each X_i is a scheme, it follows that $(X, \mathscr{F}_X)$ is a scheme as well, covered by the open subschemes $\psi_i(X_i) \cong X_i$. Moreover, for any j, the intersection $\psi_i(X_i) \cap \psi_j(X_j)$ is equal to $\psi_i(U_{ij})$, and the gluing isomorphisms satisfy $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}, completing the construction. □

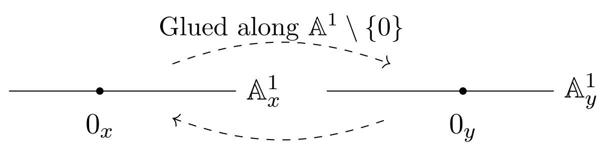
Example 18.5. (Affine line with two origins) Let $\mathbb{K}$ be an algebraically closed field. Consider the schemes

$$X_1 = \operatorname{Spec} \mathbb{K}[s],$$
$$X_2 = \operatorname{Spec} \mathbb{K}[t].$$

Define the open subschemes $U_{12} \subseteq X_1$ and $U_{21} \subseteq X_2$ by $U_{12} = U_s$ and $U_{21} = U_t$, respectively. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism of schemes induced by the ring isomorphism

$$\mathbb{K}[t, t^{-1}] \xrightarrow{\sim} \mathbb{K}[s, s^{-1}]$$

which sends $t \mapsto s$. By Proposition 18.4, this data determines a scheme obtained by gluing X_1 and X_2 along $U_{12} \cong U_{21}$. The resulting scheme is commonly referred to as the affine line with two origins.



The affine line with two origins: two copies of $\mathbb{A}^1$ glued along the complement of the origin.

Let us now discuss an example of a scheme that retains more geometric information than a classical affine algebraic set. This illustrates the power of the scheme concept.

19. Reduced, Integral and Noetherian Schemes

Although the notion of a scheme is extremely general and flexible, this generality comes at a cost: various pathologies may arise in the absence of additional structure. To obtain schemes that more closely resemble classical geometric objects, or that exhibit desirable behavior under common constructions, it is often useful to restrict attention to schemes satisfying certain structural conditions. We introduce several important classes of schemes defined by properties of their rings of sections. In particular, we consider reduced, integral, and Noetherian schemes.

19.1. **Reduced Schemes.**

Consider an element of a ring R that vanishes at every point of $\operatorname{Spec} R$; that is, an element contained in every prime ideal of R. One might initially expect such an element to be zero; however, this need not be the case. The condition of lying in all prime ideals is precisely equivalent to belonging to the nilradical of the ring, which may be nonzero[15]. In other words, thinking of elements of R as functions on $\operatorname{Spec} R$, the zero function need not be the only function that vanishes everywhere on $\operatorname{Spec} R$. This observation naturally motivates the following definition.

Definition 19.1. Let $(X, \mathscr{F}_X)$ be a scheme. We say that X is a **reduced scheme** if for every open subset $U \subseteq X$, the ring $\mathscr{F}_X(U)$ has no non-zero nilpotent elements; that is, $\mathscr{F}_X(U)$ is a reduced ring.

Remark 19.2. *Note that reduced schemes eliminate the problem mentioned at the beginning of this section. Indeed, if $(X, \mathscr{F}_X)$ is a reduced scheme $f, g \in \mathscr{F}_X(U)$ are such that $f = g \in \mathscr{F}_x$ for each $x \in U$ then $f = g$.*

Proposition 19.3. *(Hartshorne II.2.3) Let $(X, \mathscr{F}_X)$ be a scheme. $(X, \mathscr{F}_X)$ is reduced if and only if the stalk $\mathscr{F}_{X,x}$ is a reduced ring for all $x \in X$.*

[15] Geometrically, this means that a function vanishes at every point of the spectrum of a ring if and only if some power of it is zero.

Proof. Assume that $(X, \mathscr{F}_X)$ is a reduced scheme. For each point $x \in X$, the stalk is given by

$$\mathscr{F}_{X,x} = \varinjlim_{x \in U} \mathscr{F}_X(U),$$

where the limit is taken over all open neighborhoods U of x. Since each $\mathscr{F}_X(U)$ is a reduced ring by assumption, and the colimit of reduced rings is again reduced, it follows that $\mathscr{F}_{X,x}$ is a reduced ring. Conversely, assume that $\mathscr{F}_{X,x}$ is a reduced ring for all $x \in X$.

(1) Suppose that X is an affine scheme, $(X, \mathscr{F}_X) \cong (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$. Then $\mathscr{O}_{\operatorname{Spec} R,\mathfrak{p}} \cong R_{\mathfrak{p}}$ is a reduced ring for all $\mathfrak{p} \in \operatorname{Spec} R$. This implies that R is a reduced ring[16].
(2) For a general scheme $(X, \mathscr{F}_X)$, we can cover X by an open affine cover $\{U_\alpha\}_\alpha$, where each U_α is an open subset isomorphic to $\operatorname{Spec} R_\alpha$ for some ring R_α. By (1), each U_α is a reduced affine scheme. Let $U \subseteq X$ be any open subset. Since the collection $\{U_\alpha\}$ covers X, the family $\{U_\alpha \cap U\}_\alpha$ forms an open cover of U. Suppose $r \in \mathscr{F}_X(U)$ is nilpotent. Then for each α, the restriction of r to $\mathscr{F}_X(U_\alpha \cap U)$ is also nilpotent. Since $\mathscr{F}_X(U_\alpha \cap U)$ is a subring of $\mathscr{F}_X(U_\alpha)$, and each $\mathscr{F}_X(U_\alpha)$ is reduced, it follows that the restriction of r to $\mathscr{F}_X(U_\alpha \cap U)$ must be zero. By the sheaf property, this implies that $r = 0$ in $\mathscr{F}_X(U)$. Hence, $(X, \mathscr{F}_X)$ is a reduced scheme.

This completes the proof. □

Corollary 19.4. *A ring R is a reduced ring if and only if Spec R is reduced.*

Proof. This follows from Proposition 19.3. □

Example 19.5. Let $\mathbb{K}$ be a field. The following is a basic list of examples of reduced and non-reduced schemes:

(1) Since $\mathbb{K}$ is a reduced ring, $\operatorname{Spec} \mathbb{K}[x_1, \ldots, x_n]$ is a reduced scheme.
(2) $\operatorname{Spec} \mathbb{K}[x]/(x^2)$ is a non-reduced scheme. This is because $\mathbb{K}[x]/(x^2)$ is not a reduced ring: the element $\overline{x}$ is nonzero, but satisfies $\overline{x}^2 = 0$ in $\mathbb{K}[x]/(x^2)$.

Remark 19.6. *The equations $x = 0$ and $x^2 = 0$ define the same affine algebraic set in $\mathbb{A}^1$. Indeed, they both correspond to the single point 0. However, as schemes, the schemes*

$$X := \operatorname{Spec} \frac{\mathbb{K}[x]}{(x)} \quad \textit{and} \quad Y := \operatorname{Spec} \frac{\mathbb{K}[x]}{(x^2)}$$

are different. While topologically they are identical, consisting of only one point, the space of global sections of X is isomorphic to $\mathbb{K}$, whereas the space of global sections of Y is a two-dimensional $\mathbb{K}$-algebra. This illustrates how schemes retain information that is lost in the classical affine algebraic geometry.

What if a scheme is not reduced? To any scheme X, one can associate a *reduced scheme* X_{red}, which has the same underlying topological space as X, but is equipped with a morphism of schemes

$$X_{\text{red}} \to X.$$

We call X_{red} the *reduced scheme* associated with X. For example, if $X = \operatorname{Spec} R$ is affine, then

$$X_{\text{red}} := \operatorname{Spec}(R/\mathscr{N}(R)),$$

[16] This follows from a commutative algebra fact that R is reduced if and only if $R_{\mathfrak{p}}$ is reduced for all $\mathfrak{p} \in \operatorname{Spec} R$.

where $\mathscr{N}(R)$ denotes the nilradical of R. The natural projection homomorphism

$$R \to R/\mathscr{N}(R)$$

induces the corresponding morphism of schemes $X_{\text{red}} \to X$. More generally, we have the following result:

Proposition 19.7. *(Hartshorne II.2.3) Let $(X, \mathscr{F}_X)$ be a scheme. Let $\mathscr{F}_{X,\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathscr{F}_X(U)_{\text{red}}$. Then $(X, \mathscr{F}_{X,\text{red}})$ is a scheme, and there is a morphism of schemes $r : (X, \mathscr{F}_{X,\text{red}}) \to (X, \mathscr{F})$, which is a homeomorphism on the underlying topological spaces. Moreover, $(X, \mathscr{F}_{X,\text{red}})$ satisfies the following universal property: for any morphism $f : Y \to X$ of schemes with Y reduced, there exists a unique morphism*

$$\theta : Y \to X_{\text{red}}$$

such that the following diagram commutes:

$$\begin{array}{ccc} Y & \overset{\theta}{\dashrightarrow} & X_{\text{red}} \\ & \searrow^{f} & \downarrow r \\ & & X \end{array}$$

Remark 19.8. *The scheme $(X, \mathscr{F}_{X,\text{red}})$ constructed in Proposition 19.7 is the reduced scheme associated to X.*

Proof. Let $\mathscr{B}$ denote the base for the topology of X consisting of all open affine schemes. Let $\mathscr{F}_{X,\text{red}}$ be the sheaf associated to the $\mathscr{B}$-sheaf

$$U_\alpha = \operatorname{Spec} R_\alpha \mapsto R_\alpha/\mathscr{N}(R_\alpha),$$

Then $(X, \mathscr{F}_{X,\text{red}})$ is a scheme. Indeed, it suffices to observe that $\operatorname{Spec}(R_\alpha/\mathscr{N}(R_\alpha))$ is naturally homeomorphic to $\operatorname{Spec}(R_\alpha)$ for any ring R_α. Moreover, the family of morphisms defined on each open affine subset of X by the projection

$$R_\alpha \to R_\alpha/\mathscr{N}(R_\alpha)$$

satisfies Corollary 18.2, and therefore gives rise to a morphism of schemes

$$r : (X_{\text{red}}, \mathscr{F}_{X,\text{red}}) \to (X, \mathscr{F}_X),$$

as required. Given any morphism $f : Y \to X$, and for each open affine subset $U \subseteq X$, we have an induced homomorphism

$$f_U^\# : \mathscr{F}_X(U) \to \mathscr{F}_Y(f^{-1}(U)),$$

whose kernel contains the nilradical of $\mathscr{F}_X(U)$ since Y is reduced. Hence, there exists a unique morphism

$$\theta_U^\# : \mathscr{F}_{X,\text{red}}(U) \to \mathscr{F}_Y(f^{-1}(U))$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}_X(U) & & \\ \downarrow r_U^\# & \searrow^{f_U^\#} & \\ \mathscr{F}_{X,\text{red}}(U) & \xrightarrow{\theta_U^\#} & \mathscr{F}_Y(f^{-1}(U)) \end{array}$$

These maps define a unique morphism of sheaves $\theta^\# : \mathscr{F}_{X,\text{red}} \to f_*\mathscr{F}_Y$ and hence a unique morphism of schemes $\theta : Y \to X_{\text{red}}$ as required. □

19.2. **Integral Schemes.** We have seen that reduced schemes eliminate the pathological behavior caused by nilpotent elements. However, another natural condition we may wish to impose is that the scheme be "irreducible" in a global sense. An integral scheme is one that is both reduced and irreducible. This condition ensures that the structure sheaf behaves like the function field of an affine variety.

Definition 19.9. Let $(X, \mathscr{F})$ be a scheme. We say that X is an **integral scheme** if for every open subset $U \subseteq X$, the ring $\mathscr{F}_X(U)$ is an integral domain.

Proposition 19.10. *Let $(X, \mathscr{F})$ be a scheme. Then $(X, \mathscr{F})$ is an integral scheme if and only if it is both reduced and irreducible.*

Proof. An integral scheme is necessarily reduced, since integral domains contain no nonzero nilpotent elements. Moreover, if X is not irreducible, then there exist disjoint nonempty open subsets $U_1, U_2 \subseteq X$ such that $X = U_1 \cup U_2$. In this case, by the sheaf property, we have

$$\mathscr{F}_X(X) = \mathscr{F}_X(U_1 \cup U_2) = \mathscr{F}_X(U_1) \times \mathscr{F}_X(U_2),$$

which is not an integral domain. Hence, an integral scheme must be both reduced and irreducible. Conversely, assume that X is both irreducible and reduced. Let $U \subseteq X$ be an affine open subset, so that $U \cong \operatorname{Spec} R$ for some ring R. Since X is irreducible, $U = \operatorname{Spec} R$ is irreducible as a topological space (Proposition 6.15). This implies that the nilradical of R is a prime ideal (Corollary 6.6). On the other hand, since X is reduced the nilradical of R is trivial; that is, it is the zero ideal. Combining these two facts, we see that the zero ideal is prime, which means R is an integral domain. Therefore, $\mathscr{F}_X(U)$ is an integral domain, where U is an open set corresponding to an affine scheme. Since the property of being an integral domain is local and we have verified it for an arbitrary affine open subset $U \subseteq X$, it follows that X is an integral scheme. The argument is similar to the analogous argument given in Proposition 19.3. □

Corollary 19.11. *Let R be a ring. Then* $\operatorname{Spec} R$ *is an integral if and only if R is an integral domain.*

Proof. If R is an integral domain, then $\operatorname{Spec} R$ is irreducible by Corollary 6.6, and reduced by Corollary 19.4. Conversely, if $\operatorname{Spec} R$ is both irreducible and reduced, then by the definition of an integral scheme and the identification $\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$, it follows that R is an integral domain. □

Proposition 16.6 implies that an integral scheme, X, has a unique generic point ξ, which is characterized by the property that $X = \overline{\{\xi\}}$. We now that the local ring $\mathscr{F}_{X,\xi}$ of the generic point ξ of an integral scheme X is a field. Indeed, $\mathscr{F}_{X,\xi}$ is defined as the direct limit

$$\mathscr{F}_{X,\xi} = \varinjlim_{U \subseteq X} \mathscr{F}_X(U) = \varinjlim_{\substack{U \subseteq X \\ U \text{ affine}}} \mathscr{F}_X(U).$$

If $f_\xi \in \mathscr{F}_{X,\xi}$, then f_ξ is the equivalence class of a pair (U, f), where U is an open affine subset and $f \in \mathscr{F}_X(U)$. Since $\mathscr{F}_X(U)$ is an integral domain, f defines a *non-empty* distinguished open subset $U_f \subseteq U$. Now,

$$\mathcal{O}_X(U_f) = \mathscr{F}_X(U)_f,$$

and hence the pair $\left(U_f, f^{-1}\right)$ represents the element $f_\xi^{-1} \in \mathscr{F}_{X,\xi}$.

Proposition 19.12. *Let X be an integral scheme with generic point ξ.*

(1) *(Hartshorne II.3.6) Let $U = \operatorname{Spec} R$ be any open affine subset of X containing ξ. Then the restriction homomorphism*

$$\mathscr{F}_X(U) \to \mathscr{F}_{X,\xi}$$

induces an isomorphism

$$\operatorname{Frac}(R) \cong \mathscr{F}_{X,\xi}.$$

(2) *By identifying $\mathscr{F}_X(U)$ and $\mathscr{F}_{X,x}$ as subrings of $\mathscr{F}_{X,\xi}$, we have*

$$\mathscr{F}_X(U) = \bigcap_{x \in U} \mathscr{F}_{X,x}.$$

Proof. The proof is given below:

(1) The point ξ is also the generic point of U, and $\mathscr{F}_{X,\xi} = \mathscr{F}_{U,\xi}$. Observe that ξ corresponds to the zero ideal.
(2) WLOG we may assume that U is affine, say $U = \operatorname{Spec} R$. Let $\gamma \in \operatorname{Frac} R$ be contained in all the localizations $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec} R$. Let

$$I = \{a \in R \mid a\gamma \in R\}$$

Then, recalling the definition of localization, for every $\mathfrak{p}$, there exists some $a \in I \setminus \mathfrak{p}$. This implies that I is not contained in any prime ideal, and therefore $I = R$. In particular, $1 \in I$, so $\gamma \in R$.

This completes the proof. □

Definition 19.13. Let $(X, \mathscr{F})$ be an integral scheme with generic point ξ. The field $\mathscr{F}_{X,\xi}$ by $K(X)$. We call $K(X)$ the **field of rational functions** (or the function field of X) and an element of $K(X)$ is called a **rational function** on X.

Remark 19.14. *We say that $f \in K(X)$ is* regular at $x \in X$ *if $f \in \mathscr{O}_{X,x}$. Proposition 19.12 affirms that a rational function which is regular at every point of an open subset $U \subseteq X$ is contained in $\mathscr{F}_X(U)$.*

Example 19.15. If $\mathbb{K}$ is algebraically closed, then $\operatorname{Spec} \mathbb{K}[x_1, \cdots, x_n]$ is an integral affine scheme. Its generic point corresponds to the zero ideal, and its field of rational functions is $K(x_1, \cdots, x_n)$. A rational function is thus given by a quotient of two polynomials, and it is regular on the whole space if and only if it is given by a single polynomial.

19.3. **Noetherian Schemes.** In many situations, it is useful to impose finiteness conditions on schemes to ensure manageable behavior both algebraically and topologically. One such condition comes from the notion of Noetherian rings. A Noetherian scheme is a scheme that is locally built from Noetherian rings and satisfies a finiteness condition on its topology. These schemes form a broad and important class that includes most examples of interest in algebraic geometry.

Definition 19.16. Let X be a scheme.

(1) X is called a **locally Noetherian scheme** if it admits an open affine cover $\{\operatorname{Spec} R_i\}_{i \in I}$ such that each R_i is a Noetherian ring.
(2) X is called a **Noetherian scheme** if it is locally Noetherian and quasicompact; equivalently, if it admits a finite open affine cover $\{\operatorname{Spec} R_i\}_{i=1}^n$ with each R_i a Noetherian ring.

In the definition of a locally Noetherian scheme, we do not require every open affine subset of X to be the spectrum of a Noetherian ring. Thus, while it is immediate from the definition that the spectrum of a Noetherian ring is a Noetherian scheme, the converse is less obvious. Establishing this converse amounts to proving that Noetherian-ness is a *local property* of schemes.

Proposition 19.17. *A scheme X is locally Noetherian if and only if for* every *open affine subset $U = \operatorname{Spec} R$, R is a Noetherian ring. In particular, an affine scheme $X = \operatorname{Spec} R$ is a Noetherian scheme if and only if the ring R is a Noetherian ring.*

PROOF. Skipped. Details can be found in [Har13]. □

Part 4. Properties of Schemes

We now introduce various properties of schemes. We first discuss the affine communication lemma, which allows us to determine when certain properties are local for schemes. We then discuss fiber products of schemes, which provide a natural categorical framework for many constructions in algebraic geometry. We will then introduce introduce various types of morphisms of schemes. The number of such properties can become quite large—and, at times, quite intricate or even frustrating to track. To aid in navigating this growing list, the following table summarizes some key classes of morphisms discussed below and their interpretations.

Morphism Type	**Interpretation**
Quasi-compact	Preimages of compact opens are compact
Finite type	Locally modeled by finitely generated algebras over the base
Separated	Analog of the Hausdorff condition in topology
Proper	Analog of compactness condition in topology
Affine	Preimages of affine opens are affine
Finite	Affine and of Finite Type

20. Affine Communication Lemma

We discuss the notion of properties that can be verified *affine-locally*, a concept we have already encountered. For instance, in Section 19, we stated that Noetherian-ness is a local property: a scheme is locally Noetherian if and only if *every* open affine subset is the spectrum of a Noetherian ring. The Affine Communication Lemma (Proposition 20.2) generalizes this observation, allowing us to prove arbitrary properties of schemes, morphisms of schemes, etc., by verifying the property on an open affine cover and then extending it to *every* open affine subset.

Remark 20.1. *We also proved certain properties of schemes—such as being reduced or integral—by establishing them on an open cover of affine schemes and then extending the results to arbitrary open subsets of the scheme. This observation is also in the spirit of Proposition 20.2: one proves a property for open affine subschemes that cover a scheme and then extends the property to arbitrary open subsets.*

Proposition 20.2. *(Affine Communication Lemma) Let X be a scheme. Suppose $\mathscr{P}$ is a 'property' that is satisfied by an open cover $\mathcal{U}$ of X by affine schemes of X. Assume that $\mathscr{P}$ satisfies the following conditions*

(1) *$\mathscr{P}$ preserved by restriction: if $\operatorname{Spec} R \hookrightarrow X$ has property $\mathscr{P}$, then for all $f \in R$,*

$$\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$$

has property $\mathscr{P}$ as well.

(2) *$\mathscr{P}$ is preserved by finite gluing: if $R = (f_1, \ldots, f_n)$ and $\operatorname{Spec} R_{f_i} \hookrightarrow X$ has property $\mathscr{P}$ for all i, then $\operatorname{Spec} R \hookrightarrow X$ has property $\mathscr{P}$ as well.*

Then every open affine subscheme of X satisfies $\mathscr{P}$.

Before proving Proposition 20.2, we briefly explain how it is used in the theory of schemes:

(1) **Properties of Schemes:** Let $\mathscr{P}$ be a property of affine schemes that satisfies conditions (1) and (2) of Proposition 20.2. Then we may extend $\mathscr{P}$ to a property of arbitrary schemes by declaring that a scheme X satisfies $\mathscr{P}$ if and only if there exists an open affine cover $\{U_i\}$ of X such that each U_i satisfies $\mathscr{P}$. By Proposition 20.2, it then follows that every open affine subscheme of X satisfies $\mathscr{P}$. Any such property $\mathscr{P}$ is referred to as a *local property of schemes*.

(2) **Properties of Morphisms** (Local on Target): Let $\mathscr{P}$ be a property of morphisms from an arbitrary scheme to an affine scheme. Suppose that for each morphism $f : Y \to X$ of schemes, $\mathscr{P}$ satisfies conditions (1) and (2) of Proposition 20.2 when f is restricted to an affine scheme in the codomain. Then we can extend $\mathscr{P}$ to a property of morphisms on arbitrary schemes. Any such property $\mathscr{P}$ is referred to as *local on the target*.

(3) **Properties of Morphisms** (Local on Source): Let $\mathscr{P}$ be a property of morphisms from an affine scheme to an arbitrary scheme. Suppose that for each morphism $f : Y \to X$ of schemes, $\mathscr{P}$ satisfies conditions (1) and (2) of Proposition 20.2 when f is restricted to an affine scheme in the domain. Then we can extend $\mathscr{P}$ to a property of morphisms on arbitrary schemes. Any such property $\mathscr{P}$ is referred to as *local on the source*.

We first prove the following lemma:

Lemma 20.3. *Let $(X, \mathscr{F}_X)$ be a scheme, and let $\operatorname{Spec} R$ and $\operatorname{Spec} S$ be affine open subschemes. The intersection $\operatorname{Spec} R \cap \operatorname{Spec} S$ is a union of open subsets that are distinguished open subsets of both $\operatorname{Spec} R$ and $\operatorname{Spec} S$.*

Proof. Suppose $\mathfrak{p} \in \operatorname{Spec} R \cap \operatorname{Spec} S$. Since $\mathfrak{p}$ is contained in basic open sets in $\operatorname{Spec} R$ and $\operatorname{Spec} S$, there exist $f \in R$ and $g \in S$ such that

$$\mathfrak{p} \in \operatorname{Spec} S_g, \operatorname{Spec} R_f \subseteq \operatorname{Spec} R \cap \operatorname{Spec} S.$$

We now claim that $\operatorname{Spec} S_g$ is a distinguished open subset of $\operatorname{Spec} R$. Restriction defines a morphism

$$S = \mathscr{F}_X(\operatorname{Spec} S) \to \mathscr{F}_X(\operatorname{Spec} R_f) = R_f.$$

Let $\widetilde{g}$ be the image of g under this map. Then the points of $\operatorname{Spec} R_f$ where g vanishes coincide with the points where $\widetilde{g}$ vanishes. Hence,

$$\operatorname{Spec} S_g = \operatorname{Spec}\left((R_f)_{\widetilde{g}}\right) = \operatorname{Spec} R_{f\widetilde{g}}.$$

This completes the proof. □

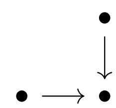
Remark 20.4. *One way to interpret the above argument is that a distinguished open subset of a distinguished open subset is itself distinguished in the original scheme.*

Proof. (Proposition 20.2) Suppose $\operatorname{Spec} R \subseteq X$ is an open affine subscheme. By assumption, there exists an open cover $\mathcal{U} = \{\operatorname{Spec} S_\alpha\}_\alpha$ of X by affine open subschemes such that each $\operatorname{Spec} S_i$ satisfies $\mathscr{P}$. By Lemma 20.3, the intersection $\operatorname{Spec} S_\alpha \cap \operatorname{Spec} R$ can be covered by open subsets which are distinguished open subsets of $\operatorname{Spec} R$ (and hence of X). By (1), each of these distinguished open sets satisfies $\mathscr{P}$. Therefore, $\operatorname{Spec} R$ admits a cover by distinguished open subsets $\operatorname{Spec} R_f$ that satisfy $\mathscr{P}$. Since $\operatorname{Spec} R$ is quasi-compact, we may extract a finite subcover, and by (2) it follows that $\operatorname{Spec} R$ itself satisfies $\mathscr{P}$. □

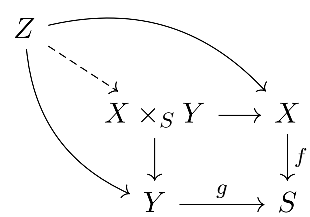
21. Fiber Products

The fiber product of schemes is a fundamental construction that generalizes the Cartesian product to the category of schemes over a base. It enables one to pull back schemes along morphisms, allowing the comparison and manipulation of families of schemes in a base-compatible way. It also allows us to unambiguously define product of affine algebraic sets and plays a central role in defining families of schemes. In what follows, let S be a fixed scheme. We first recall the definition of the fiber product in the context of schemes over S.

Definition 21.1. Let X, Y be schemes over S. The fiber product of X and Y, denoted as $X \times_S Y$, is the product in Sch_S. That is, if I is the small category,



then $X \times_S Y$, is a terminal object fitting into the diagram:



Example 21.2. Since $\operatorname{Spec}\mathbb{Z}$ is a final object in Sch, it follows by categorical considerations [17] that the fiber product over $\operatorname{Spec}\mathbb{Z}$ coincides with the usual product:

$$X \times_{\operatorname{Spec}\mathbb{Z}} Y = X \times Y.$$

A priori, it is not clear that the fiber product of schemes exists. However, if it does exist, then it is unique up to isomorphism by standard category-theoretic arguments. We now argue that the fiber product indeed exists. As usual, our strategy is to proceed in a bottom-up manner, by first establishing the result in the case of affine schemes.

Lemma 21.3. *Let X, Y be affine S-scheme where S is an affine scheme. Then $X \times_S Y$ exists.*

Proof. Assume that

$$X \cong \operatorname{Spec} R$$
$$Y \cong \operatorname{Spec} S$$
$$S \cong \operatorname{Spec} T$$

Recall that the pushout in CRing is the tensor product. Since $\mathsf{AffSch} \cong \mathsf{CRing}^{\mathrm{Op}}$, we can immediately conclude by applying the Spec functor

$$\operatorname{Spec} R \times_{\operatorname{Spec} T} \operatorname{Spec} S \cong \operatorname{Spec}(R \otimes_T S)$$

This completes the proof. □

Example 21.4. The following is a basic list of computations of fiber products of affine schemes:

[17] If S is a final object in a category C, then for any two objects $X, Y \in \mathsf{C}$, we have $X \times_S Y \cong X \times Y$.

(1) Let $\mathbb{K}$ be a field. Let $X = \operatorname{Spec}(\mathbb{K}[x])$ and $Y = \operatorname{Spec}(\mathbb{K}[y])$ be affine schemes over $\mathbb{K}$. We have

$$X \times_{\operatorname{Spec}\mathbb{K}} Y \cong \operatorname{Spec}(\mathbb{K}[x] \otimes_{\mathbb{K}} \mathbb{K}[y]) \cong \operatorname{Spec}(\mathbb{K}[x,y])$$

(2) Let $\mathbb{K}$ be a field. More generally, let $X = \operatorname{Spec}(\mathbb{K}[x_1, \cdots, x_n]) := \mathbb{A}^n$ and $Y = \operatorname{Spec}(\mathbb{K}[y_1, \cdots, y_m]) := \mathbb{A}^m$ be affine schemes over $\mathbb{K}$. We have

$$\begin{aligned} \mathbb{A}^n \times_{\operatorname{Spec}\mathbb{K}} \mathbb{A}^m &= X \times Y \\ &\cong \operatorname{Spec}\left(\mathbb{K}[x_1, \ldots, x_n] \otimes_{\mathbb{K}} \mathbb{K}[y_1, \ldots, y_m]\right) \\ &\cong \operatorname{Spec}\left(\mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]\right) \\ &\cong \mathbb{A}^{n+m}. \end{aligned}$$

Hence, in the case of affine varieties over a field $\mathbb{K}$ the fiber product agrees with the geometric intuition: the product of affine n-space and affine m-space over $\mathbb{K}$ is affine $(n+m)$-space over $\mathbb{K}$. Thus, the fiber product construction recovers expected geometric behavior in a categorical framework.

(3) Let $\mathbb{K}$ be a field.

$$V = \operatorname{Spec}\left(\frac{\mathbb{K}[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)}\right) \subseteq \mathbb{A}^n \quad \text{and} \quad W = \operatorname{Spec}\left(\frac{\mathbb{K}[y_1, \ldots, y_r]}{(g_1, \ldots, g_s)}\right) \subseteq \mathbb{A}^r$$

The fiber product $V \times_{\operatorname{Soec}\mathbb{K}} W$ is the affine algebraic set in $\mathbb{A}^{n+r}$ cut out by the vanishing of the polynomials $f_1, \ldots, f_m, g_1, \ldots, g_s$ where the variables x_i and y_j are considered to be independent. That is,

$$V \times_{\operatorname{Spec}\mathbb{K}} W = \operatorname{Spec}\left(\frac{\mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_r]}{(f_1, \ldots, f_m, g_1, \ldots, g_s)}\right) \subseteq \mathbb{A}^{n+r}.$$

(4) The fiber product over $\operatorname{Spec}\mathbb{Z}$ of two nonzero affine schemes may be the zero scheme:

$$\operatorname{Spec}\mathbb{Z}/m \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}/n \cong \operatorname{Spec}\left(\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n\right) = \operatorname{Spec}\mathbb{Z}/(m,n).$$

When $(m,n) = 1$, we have $\mathbb{Z}/(m,n) \cong 0$, so the fiber product is the empty scheme. However, if we instead work in the category of schemes over a field $\mathbb{K}$, such issues do not arise. Since any $\mathbb{K}$-algebra contains a copy of $\mathbb{K}$, the tensor product of two nonzero $\mathbb{K}$-algebras is nonzero. Consequently, the fiber product of two nonempty schemes over $\mathbb{K}$ is always non-empty.

More generally, we have the following result:

Proposition 21.5. *Let X, Y be schemes over a scheme X. Then $X \times_S Y$ exists and is unique up to unique isomorphism.*

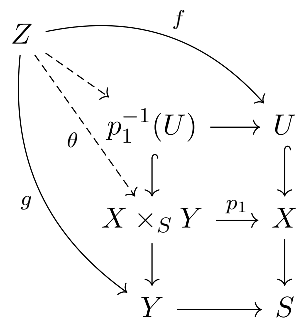
Since schemes are constructed by gluing affine schemes along open subsets, the existence of fiber products of schemes should follow from the existence of fiber products of affine schemes over affine bases (Lemma 21.3). The gluing mechanism should then allow us to construct fiber products of arbitrary schemes over arbitrary base schemes. The argument will repeatedly rely on the same guiding principle: schemes glue, and morphisms of schemes glue.

Proof. (Proposition 21.5) Uniqueness is clear.

(1) We first argue that if X and Y are schemes over a base scheme S, and if $U \subseteq X$ is an open subset, then whenever the fiber product $X \times_S Y$ exists, the fiber product $U \times_S Y$ also exists and is given by the open subset

$$p_1^{-1}(U) \subseteq X \times_S Y,$$

where $p_1 : X \times_S Y \to X$ is the projection such that the following diagram commutes:



We verify that $p_1^{-1}(U)$ satisfies the universal property of the fiber product $U \times_S Y$. Let Z be a scheme and suppose we are given morphisms f, g as shown in the diagram above. The composition $U \hookrightarrow X$ followed by f gives a morphism $Z \to X$ compatible with g over S. Then by the universal property of $X \times_S Y$, there exists a unique morphism $\theta : Z \to X \times_S Y$ making the diagram commute. Since $f(Z) \subseteq U$, the image of θ is contained in $p_1^{-1}(U) \subseteq X \times_S Y$, so θ factors uniquely through $p_1^{-1}(U)$, giving a morphism $Z \to p_1^{-1}(U)$. This proves that $p_1^{-1}(U)$ satisfies the universal property and thus realizes the fiber product $U \times_S Y$.

(2) We now argue that if X and Y are schemes over a base scheme S, and if $\{X_j\}_{j \in I}$ is an open cover of X such that for each j, the fiber product $X_j \times_S Y$ exists, then the fiber product $X \times_S Y$ also exists. Indeed, for each pair $i, j \in I$, define

$$U_{ij} := p_i^{-1}(X_i \cap X_j) \subseteq X_i \times_S Y,$$

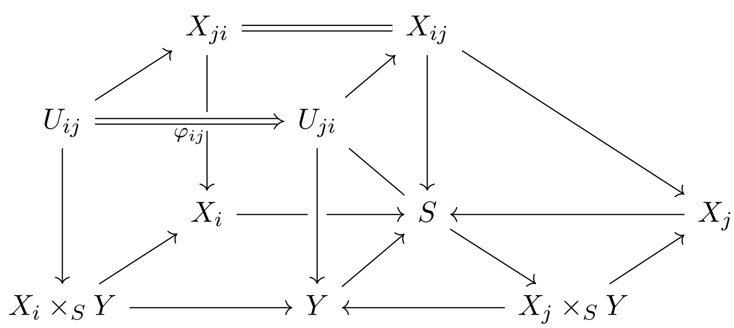
where $p_i : X_i \times_S Y \to X_i$ is the projection. Then U_{ij} is a fiber product for $X_i \cap X_j$ and Y over S, which exists by (1). By the universal property of fiber products, and in particular their uniqueness up to unique isomorphism, for each pair i, j there is a unique isomorphism

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

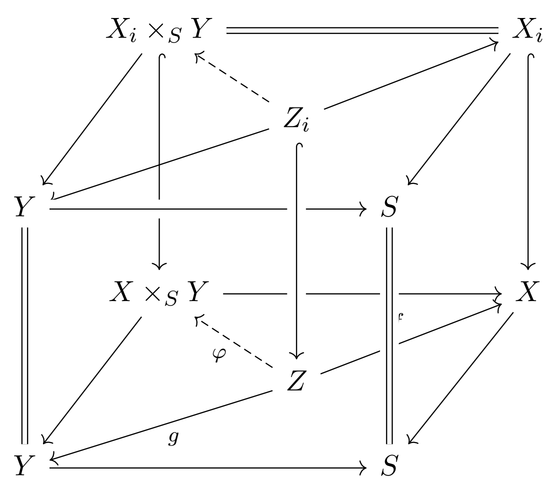
that is compatible with all structure morphisms and projections. These isomorphisms satisfy the cocycle condition[18] and therefore allow us to glue the $\{X_j \times_S$

[18]We skip the details checking that the morphisms φ_{ij} satisfy the hypothesis of the theorem that allows us to glue scheames. We refer the reader to see this link for the relevant diagram chasing details.

$Y\}_{j\in J}$ along the $\{U_{ij}\}_{i,j\in J}$ to obtain a scheme $X \times_S Y$, making the following diagram commute:



The projection morphisms p_1 and p_2 are defined by gluing the projections from the pieces $X_i \times_S Y$. Given a scheme Z and morphisms $f : Z \to X$, $g : Z \to Y$, let $Z_i = f^{-1}(X_i)$. Then we get maps $\theta_i : Z_i \to X_i \times_S Y$, hence, by composition with the inclusions $X_i \times_S Y \subset X \times_S Y$, we get maps $\phi_j : Z_i \to X \times_S Y$. These maps agree on intersections $Z_i \cap Z_j$, so we can glue the morphisms to obtain a morphism $\phi : Z \to X \times_S Y$, compatible with the projections and f and g. Uniqueness is clear.



(3) We have established that the fiber product of affine schemes exists. It follows that for any scheme X and affine schemes Y and S, the fiber product $X \times_S Y$ exists. By symmetry, interchanging the roles of X and Y, we conclude that the fiber product exists for any schemes X and Y over an affine base scheme S.

(4) Given arbitrary schemes X, Y, and S, let $q : X \to S$ and $r : Y \to S$ be the given morphisms. Let $\{S_j\}$ be an open affine cover of S, and define $X_j = q^{-1}(S_j)$ and $Y_j = r^{-1}(S_j)$. Then $X_j \times_{S_j} Y_j$ exists, since the fiber product of schemes over an affine base exists. Moreover, this same scheme serves as a fiber product for X_j and Y over S. Indeed, given morphisms $f : Z \to X_j$ and $g : Z \to Y$ over S, the image of Z under the composite $Z \xrightarrow{g} Y \xrightarrow{r} S$ must lie in S_j, and hence the image of Z under g lies in Y_j. Thus, $X_j \times_S Y$ exists for each j. Applying the same gluing argument again, we conclude that the fiber product $X \times_S Y$ exists.

This completes the proof. □

Remark 21.6. *If S is a fixed scheme, then in the category of schemes over S, the scheme S itself is the final object. In this context, categorical considerations imply that*

$$X \times_S S \cong X.$$

More generally, for any two S-schemes X and Y, we recover the usual product of schemes when the base is a final object:

$$X \times_S Y \cong X \times Y.$$

We now turn to discussing various applications and examples of fiber products. We begin by introducing the notion of base change/extension. Using this concept, we proceed to compute fiber products of affine schemes in several illustrative cases.

21.1. **Base Change/Extension.** Let S be a fixed scheme, which we regard as the base scheme. If S' is another scheme equipped with a morphism $S' \to S$, then for any scheme X over S, we define the *base change* of X along $S' \to S$ as the fiber product

$$X' := X \times_S S',$$

which is naturally a S'-scheme. For example, if $S = \operatorname{Spec} \mathbb{K}$ for a field $\mathbb{K}$, and $S' = \operatorname{Spec} \mathbb{K}'$ where $\mathbb{K}'$ is a field extension of $\mathbb{K}$, then X' is the base extension of X to the larger field $\mathbb{K}'$.

$$\begin{array}{ccc} X \times_{\operatorname{Spec} \mathbb{K}} \operatorname{Spec} \mathbb{K}' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{K}' & \longrightarrow & \operatorname{Spec} \mathbb{K} \end{array}$$

Example 21.7. Consider the $\mathbb{R}$-valued points of the circle defined by the scheme

$$\operatorname{Spec}\big(\mathbb{R}[x,y]/(x^2+y^2-1)\big).$$

If we wish to instead consider the $\mathbb{C}$-valued points, observe that

$$\operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2-1))$$

is a $\operatorname{Spec} \mathbb{R}$-scheme, and $\operatorname{Spec} \mathbb{C}$ is a $\operatorname{Spec} \mathbb{R}$-scheme. The base extension (fiber product) is then

$$\begin{aligned} \operatorname{Spec}\big(\mathbb{R}[x,y]/(x^2+y^2-1)\big) \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} &\cong \operatorname{Spec}\big(\mathbb{R}[x,y]/(x^2+y^2-1) \otimes_{\mathbb{R}} \mathbb{C}\big) \\ &\cong \operatorname{Spec}\big(\mathbb{C}[x,y]/(x^2+y^2-1)\big). \end{aligned}$$

We now use the machinery of base extension to develop tools for computing fiber products in practice.

(1) Suppose S is a subring of R. It is a basic algebraic fact that

$$R \otimes_S S[t] \cong R[t].$$

As a consequence, we have an isomorphism of schemes:

$$\operatorname{Spec} R[t] \cong \operatorname{Spec} R \times_{\operatorname{Spec} S} \operatorname{Spec} S[t].$$

Thus, the following diagram is a fiber product square, and $\operatorname{Spec} R[t]$ is the fiber product obtained by *adding extra variables*:

$$\begin{array}{ccc} \operatorname{Spec} R[t] & \longrightarrow & \operatorname{Spec} S[t] \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} S. \end{array}$$

(2) Suppose $\varphi : S \to R$ is a ring homomorphism, and let $I \subseteq S$ be an ideal. It is a basic algebraic fact that the extension of I to R, denoted by $I^e = I \cdot R$, satisfies

$$\frac{R}{I^e} \cong R \otimes_S \frac{S}{I}.$$

As a consequence, there is an isomorphism of schemes

$$\operatorname{Spec}\left(\frac{R}{I^e}\right) \cong \operatorname{Spec} R \times_{\operatorname{Spec} S} \operatorname{Spec}\left(\frac{S}{I}\right).$$

For example, we compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$:

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/(x^2+1)) \\ &\cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x])/(x^2+1) \\ &\cong \mathbb{C}[x])/(x^2+1) \\ &\cong \mathbb{C}[x])/(x-i) \times \mathbb{C}[x])/(x+i) \cong \mathbb{C} \times \mathbb{C} \end{aligned}$$

As a result:

$$\operatorname{Spec}(\mathbb{C} \times_{\mathbb{R}} \mathbb{C}) \cong \operatorname{Spec} \mathbb{C} \coprod \operatorname{Spec} \mathbb{C}$$

(3) Consider the *base change of affine schemes by localization*. Suppose $\phi : B \to A$ is a ring homomorphism, and let $S \subseteq B$ be a multiplicative subset. Since $\phi(S) \subseteq A$ is also multiplicative, it is an algebraic fact that

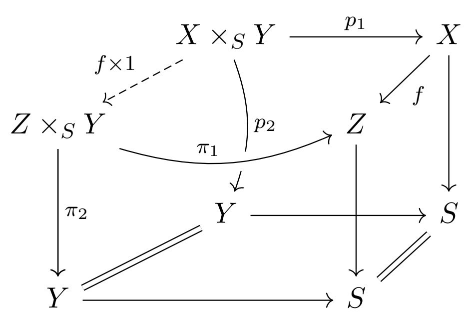
$$\phi(S)^{-1} A \cong A \otimes_B S^{-1} B.$$

As a consequence, we have an isomorphism of schemes:

$$\operatorname{Spec}\left(\phi(S)^{-1} A\right) \cong \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} S^{-1} B.$$

We say that localizations is preserved by base change,

21.2. **Product of Morphisms.** Let $f : X \to Z$ and $g : Y \to H$ be morphisms of S-schemes. Consider the following commutative diagram:

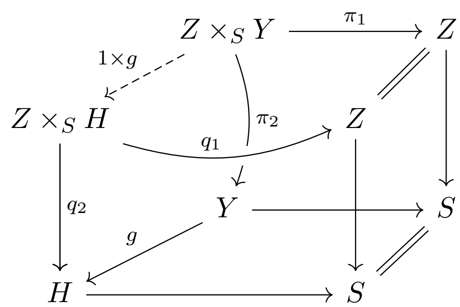


The front and back faces of the cube are cartesian squares denoting base change. The morphisms $f \circ p_1$ and $1_Y \circ p_2$ form a commutative diagram with the morphisms $Z \to S$

and $H \to S$. Hence, by the universal property of the fiber product $Z \times_S H$, there exists a unique morphism

$$f \times 1 : X \times_S Y \to Z \times_S H$$

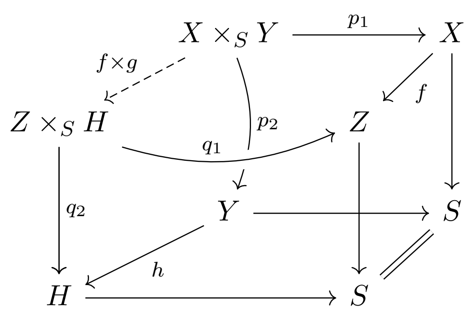
making the above diagram commute. Similarly, consider the following diagram:



The front and back faces of the cube are cartesian squares denoting base change. The morphisms $1_Z \circ \pi_1$ and $g \circ \pi_2$ form a commutative diagram with the morphisms $Z \to S$ and $H \to S$. Hence, by the universal property of the fiber product $Z \times_S H$, there exists a unique morphism

$$1 \times g : Z \times_S Y \to Z \times_S H$$

making the above diagram commute.



Composing the two morphisms above yields the product morphism:

$$f \times g : X \times_S Y \to Z \times_S H.$$

21.3. **Fibers of Morphisms.** An important application of fiber products is the definition of fibers of a morphism. Given a morphism of schemes $f : X \to Y$ and a point $y \in Y$, the fiber of f over y is defined as the fiber product

$$X_y := X \times_Y \operatorname{Spec}(\kappa(y)),$$

where $\kappa(y) = \mathscr{F}_{Y,y}/\mathfrak{m}_y$ denotes the residue field at the point y. This construction captures the geometric intuition of 'the set of points in X lying over y,' but in a way that is compatible with the structure of schemes.

Remark 21.8. *Note that $X \times_Y \operatorname{Spec}(\kappa(y))$ is well-defined because there exists a morphism $\operatorname{Spec}(\kappa(y)) \to Y$ defined as the composition of the following two morphisms:*

(1) *The natural inclusion*

$$\operatorname{Spec}(\kappa(y)) \to \operatorname{Spec}(\mathscr{F}_{Y,y}),$$

induced by the quotient map $\mathscr{F}_{Y,y} \to \mathscr{F}_{Y,y}/\mathfrak{m}_y = \kappa(y)$.

(2) *The canonical map*

$$\operatorname{Spec}(\mathscr{F}_{Y,y}) \to Y,$$

which is defined by the structure of Y *as a locally ringed space. This morphism is constructed as follows: any point* $y \in Y$ *determines a local ring* $\mathscr{F}_{Y,y}$*, and for every affine open neighborhood* $U = \operatorname{Spec} R \subseteq Y$ *containing* y*, there is a natural ring homomorphism* $A \to \mathscr{F}_{Y,y}$ *given by localization. These homomorphisms define a compatible system of morphisms of schemes*

$$\operatorname{Spec}(\mathscr{F}_{Y,y}) \to \operatorname{Spec} R \hookrightarrow Y,$$

and by gluing, this yields a morphism $\operatorname{Spec}(\mathscr{F}_{Y,y}) \to Y$*. Intuitively, this map sends each prime ideal* $\mathfrak{p} \subseteq \mathscr{F}_{Y,y}$ *to the unique point of* Y *whose stalk maps to* $\mathfrak{p}$ *under the localization maps.*

Fibers play a central role in many aspects of algebraic geometry, including the study of how the geometry of X varies with $y \in Y$. We want to prove that this definition coincides with the purely topological one.

Proposition 21.9. *(Hartshorne II.3.10) Let* X, Y *be schemes. If* $f : X \to Y$ *is a morphism of schemes, and* $y \in Y$ *is a point, then the topological space underlying* X_y *is homeomorphic to* $f^{-1}(y)$ *with the induced topology.*

Proof. It suffices to consider the case of affine schemes. The general case then follows by means of open affine coverings. Assume that $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$. Identify the point $y \in Y$ with a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$. In this setting, the fiber product of schemes corresponds to the following diagram of commutative rings:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ \operatorname{Frac}(R/\mathfrak{p}) & \longrightarrow & S \otimes_R \operatorname{Frac}(R/\mathfrak{p}) \end{array}$$

Moreover, we have the following identification of the fiber over y:

$$\begin{aligned} X_y &:= X \times_Y \operatorname{Spec} \kappa(y) \\ &= \operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \\ &= \operatorname{Spec}\left(S \otimes_R R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}\right) \\ &\cong \operatorname{Spec}\left(S^{-1}(S/\mathfrak{p}^e)\right), \end{aligned}$$

where $S = \varphi(R \backslash \mathfrak{p})$ is the multiplicatively closed subset of S consisting of images of elements in $R \setminus \mathfrak{p}$, and $\mathfrak{p}^e$ denotes the extension of the ideal $\mathfrak{p}$ in S via φ. Finally, the set-theoretic fiber

$$f^{-1}(y) = \{\mathfrak{q} \in \operatorname{Spec} S \mid \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\}$$

consists precisely of those prime ideals $\mathfrak{q} \subseteq S$ such that $\mathfrak{q} \cap S = \emptyset$ and $\mathfrak{p}^e \subseteq \mathfrak{q}$. This set corresponds exactly to

$$\operatorname{Spec}\left(S^{-1}(S/\mathfrak{p}^e)\right).$$

This completes the proof. □

Example 21.10. We compute $\operatorname{Spec}\mathbb{Z}[x]$. The inclusion $\mathbb{Z} \to \mathbb{Z}[x]$ induces a morphism

$$\operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}.$$

To understand the prime ideals of $\operatorname{Spec}\mathbb{Z}[x]$, it suffices to analyze the fibres of this morphism. The fibre over the prime ideal $\langle 0 \rangle$ is given by

$$\operatorname{Spec}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[x]) = \operatorname{Spec}\mathbb{Q}[x],$$

which corresponds to all irreducible polynomials over $\mathbb{Q}$ together with the zero ideal. Similarly, the fibre over $\langle p \rangle$ is

$$\operatorname{Spec}\mathbb{F}_p[x],$$

which consists of the irreducible polynomials over $\mathbb{F}_p$ along with its zero ideal. (Note that the zero ideals correspond to those in $\mathbb{Z}$.)

22. Finite Type Morphisms

We discuss morphisms of schemes that exhibit finiteness properties, with a primary focus on those of finite type. Such morphisms play a central role in algebraic geometry, as they generalize classical notions of algebraic finiteness and provide control over the complexity of morphisms between schemes.

22.1. **Quasicompact Morphisms.**

Before discussing morphisms of finite type, it is important to first understand quasicompact morphisms, as they form a foundational finiteness condition. Many properties of finite type morphisms build upon or refine the notion of quasicompactness.

Definition 22.1. Let X, Y be schemes. A morphism $f : X \to Y$ of schemes is **quasicompact** if the preimage of every open affine subscheme of Y is a compact set in X.

Quasicompact morphisms are particularly tractable, as finite open coverings are generally easier to handle in practice than infinite ones. Moreover, many schemes that arise in geometric or arithmetic applications are quasicompact.

Lemma 22.2. *Let X, Y be schemes. A morphism $f : X \to Y$ of schemes is quasicompact if and only if the preimage of every compact open subset of Y is compact subset in X.*

Proof. Suppose f is quasicompact. Then any compact open subset $U \subseteq Y$ can be covered by finitely many affine open subsets $\{V_i\}_{i=1}^n$. Since f is quasicompact, $f^{-1}(V_i)$ is compact for each $i = 1, \cdots, n$. Therefore, the pre-image $f^{-1}(U) = \bigcup_{i=1}^n f^{-1}(V_i)$ is a finite union of compact open subsets, and hence compact. The converse follows from the fact that an affine scheme is compact. □

Proposition 22.3. *The following is a list of properties of quasicompact morphims.*

1. A composition of two quasicompact morphisms is quasicompact.
2. Any morphism from a Noetherian scheme is quasicompact.
3. (*Hartshorne II.3.2)* A morphism $f : X \to Y$ is quasicompact iff there is a cover of Y by affine open subsets $\{U_i\}_{i \in I}$ such that $f^{-1}(U_i)$ is a compact subset.

Proof. The proof is given below:

1. This follows immediately from Lemma 22.2.
2. This follows because every open subset of a Noetherian topological space is itself compact.

(3) The forward direction is immediate. For the reverse direction, we apply the Affine Communication Lemma (Proposition 20.2).
   (a) Suppose that
$$f^{-1}(\operatorname{Spec} S) = \bigcup_{i=1}^{n} \operatorname{Spec} R_i$$
   is a finite affine open cover, so that $f^{-1}(\operatorname{Spec} S)$ is compact. Then for any $s \in S$, we have
$$f^{-1}(\operatorname{Spec} S_s) = \bigcup_{i=1}^{n} \operatorname{Spec}(R_i)_{r_i},$$
   where each $r_i \in R_i$ is the image of s under the induced morphism $S \to R_i$ corresponding to the restriction of f to $\operatorname{Spec} R_i$. Since localization preserves compactness of affine open sets, each $\operatorname{Spec}(R_i)_{r_i}$ is compact, and so $f^{-1}(\operatorname{Spec} S_s)$ is compact as a finite union of compact open sets.
   (b) Suppose
$$\operatorname{Spec} S = \bigcup_{j=1}^{n} \operatorname{Spec} S_{s_j}$$
   is an open affine cover, and that each preimage $f^{-1}(\operatorname{Spec} S_{s_j})$ is quasicompact. Then
$$f^{-1}(\operatorname{Spec} S) = \bigcup_{j=1}^{n} f^{-1}(\operatorname{Spec} S_{s_j})$$
   is a finite union of compact spaces, and hence itself a compact set.

   Both conditions of the Affine Communication Lemma (Proposition 20.2) are satisfied, and the claim follows.

This completes the proof. □

22.2. **Finite Type Morphisms.** Recall that an affine algebraic set is the spectrum of a finitely generated $\mathbb{K}$-algebra. To describe this situation for more general schemes we make a general definition concerning morphisms of schemes.

Definition 22.4. Let X, Y be schemes and $f : X \to Y$ be a morphism of schemes.

(1) f is of **locally finite type** if there exists an open covering $\{U_i\}_i$ of Y by affine schemes $U_i = \operatorname{Spec} S_i$ such that the preimage $V_{ij} := f^{-1}(U_i)$ can be covered by open affine subsets $\{\operatorname{Spec} R_{ij}\}_{i,j}$ with the property that the induced ring homomorphisms $S_i \to R_{ij}$ make each R_{ij} a finitely generated S_i-algebra.
(2) f is of **finite type** if, in addition, each $f^{-1}(U_i)$ can be covered by finitely many such affine subsets V_{ij}.

Remark 22.5. *In both cases, the associated morphism of rings is said to be finite, and the target ring is a finitely generated algebra over the source ring.*

Observe that the property of the morphism f is defined in terms of an open affine covering of the target scheme Y. This reflects a common paradigm: a property $\mathcal{P}$, initially formulated for morphisms between affine schemes, is required to hold only locally on Y, that is, on an open affine cover. The equivalence between verifying the property $\mathcal{P}$ globally on Y and verifying it on such a covering is guaranteed by the Affine Communication Lemma (see Proposition 20.2).

Proposition 22.6. (Hartshorne II.3.1) *Let X, Y be schemes. A morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} S \subseteq Y$ and every open affine subset $U = \operatorname{Spec} R \subseteq f^{-1}(V)$, R is a finitely generated S-algebra.*

Proof. The reverse implication is straightforward. For the forward direction, we apply the Affine Communication Lemma (Proposition 20.2). Fix an affine open subset $V = \operatorname{Spec} S \subseteq Y$. Without loss of generality, assume $Y = \operatorname{Spec} S$.

(1) Assume that
$$f^{-1}(\operatorname{Spec} S) = \bigcup_{i \in I} \operatorname{Spec} R_i,$$
where each R_i is a finitely generated S-algebra. That is, for each i, there exist elements $r_{ij_1}, \ldots, r_{ij_l} \in R_i$ such that
$$R_i = S[r_{ij_1}, \ldots, r_{ij_l}].$$
Equivalently, each R_i is a finitely generated algebra over S via a ring homomorphism $\varphi_i : S \to R_i$. Fix an element $s \in S$. Since each R_i is a unital S-algebra, we have
$$\varphi_i(s) = s \cdot 1_{R_i} \in R_i.$$
For any $s \in S$, note that
$$f^{-1}(\operatorname{Spec} S_s) = \bigcup_{i \in I} \operatorname{Spec}(R_i)_{\varphi_i(s)}.$$
Moreover, for any prime ideal $\mathfrak{p} \subseteq S$ such that $s \notin \mathfrak{p}$, the fiber $f^{-1}(\mathfrak{p})$ corresponds to a prime ideal $\mathfrak{q} \subseteq R_i$ for some i, and hence $\varphi_i(s) \notin \mathfrak{q}$. This shows that the localization $(R_i)_{\varphi_i(s)}$ is defined and is a finitely generated algebra over S_s.

(2) Assume that there exist elements $s_1, \ldots, s_n \in S$ such that the ideal generated by these elements is the unit ideal, i.e. $(s_1, \ldots, s_n) = S$, and for each j,
$$f^{-1}(\operatorname{Spec} S_{s_j}) = \bigcup_{i \in I} \operatorname{Spec} R_i,$$
where each R_i is a finitely generated S_{s_j}-algebra. Since
$$\operatorname{Spec} S = \bigcup_{j=1}^{n} U_{s_j} = \bigcup_{j=1}^{n} \operatorname{Spec} S_{s_j},$$
we have
$$f^{-1}(\operatorname{Spec} S) = \bigcup_{j=1}^{n} f^{-1}(\operatorname{Spec} S_{s_j}) = \bigcup_{j=1}^{n} \bigcup_{i \in I} \operatorname{Spec} R_i.$$
Because each R_i is finitely generated over S_{s_j} and the s_j's generate the unit ideal in S, it follows by a standard argument (clearing denominators) that each R_i is finitely generated over S.

Since both conditions hold, the Affine Communication Lemma (Proposition 20.2) implies the desired result. □

We discuss some properties of (locally) finite type morphisms below.

Proposition 22.7. *(Hartshorne II.3.3) The following is a is a list of properties of (locally) finite type morphisms.*

(1) A morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.
(2) A composition of two morphisms (locally) of finite type is (locally) of finite type.
(3) An open immersion is locally of finite type.
(4) A quasicompact open immersion is of finite type.
(5) An open immersion into a locally Noetherian scheme is of finite type.
(6) If $f : X \to Y$ is (locally) of finite type, and Y is locally Noetherian, then X is (locally) Noetherian.
(7) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is of finite type and f is quasi-compact, then f is of finite type.
(8) If X is a scheme of finite type over $\mathbb{K}$, then there is a finite covering of X consisting of open affine subsets that are spectra of finitely generated $\mathbb{K}$-algebras.
(9) Morphisms of finite type are stable under base extension.
(10) If X and Y are S-schemes of finite type, then $X \times_S Y$ is a S-scheme of finite type.

Proof. The proof is given below:

(1) The forward direction is clear. Conversely, if f is locally of finite type and quasi-compact, then each $f^{-1}(V_i)$ is quasi-compact, so we may find a finite subcover, making it of finite type.
(2) This essentially follows from the algebraic fact that a composition of finite ring maps is finite.
(3) Let $V = \operatorname{Spec} S$ be such an affine open subset in Y. The map $f^{-1}(V) \to V$ is an open immersion. Hence, $f^{-1}(V) \cong V = \operatorname{Spec} S$. Cover V by open subsets $\{\operatorname{Spec} S_f \mid f \in S\}$, and note that each

$$S_f = S[x]/(xf - 1)$$

is a finitely generated S-algebra.
(4) This follows from (3).
(5) This follows from the algebraic fact that a ring of finite type over a Noetherian ring is a Noetherian ring.
(6) This follows from the algebraic fact that that a ring of finite type over a Noetherian ring is Noetherian.
(7) The affine case essentially follows from the fact that

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

is a composition of morphisms of rings such that C is a finitely generated A-algebra, then C is a finitely generated B-algebra. The general case reduces to the affine case[19]
(8) This is clear.

[19] If the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ is of finite type, we need to find a covering of Y consisting of open affine sets with the desired property. This is the case if we can find such a covering for $g^{-1}(U)$ for any open affine subset $U \subseteq Z$. Hence, we have reduced the problem to proving the statement assuming $Z = \operatorname{Spec} C$ is affine. Since f is quasi-compact, for any open affine subset $V = \operatorname{Spec} B$ of Y, we can cover $f^{-1}(V)$ with a finite number of open affine subsets $\operatorname{Spec} A_i$ of X. Since the composition $g \circ f$ is of finite type, each ring A_i is a finitely generated C-algebra, and we are again in the affine case.

(9) This Let $f : X \to S$ be any morphism of finite type, and let $g : S' \to S$ be any base extension. Then, construct the pull-back of f along g:

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

Let $U \subseteq S$ be an open affine set, say $U = \operatorname{Spec} R$. Then the diagram restricts to the following:

$$\begin{array}{ccc} f^{-1}(U) \times_U g^{-1}(U) & \xrightarrow{p_1} & f^{-1}(U) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ g^{-1}(U) & \xrightarrow[g]{} & U \end{array}$$

For each $V \subseteq g^{-1}(U)$ open affine set, say $V = \operatorname{Spec} S$, we can shrink more and get the following:

$$\begin{array}{ccc} f^{-1}(U) \times_U V & \xrightarrow{p_1} & f^{-1}(U) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[g]{} & U \end{array}$$

There is a finite covering of $f^{-1}(U)$ by affine open subsets of the form $\operatorname{Spec} T_i$ with T_i a finitely generated R-algebra. Hence $f^{-1}(U) \times_U V$ can be covered by open affine sets of the form $\operatorname{Spec}(T_i \otimes_R S)$, and each one of these rings is a finitely generated S-algebra.

(10) This follows from by reducing to the affine case and using the algebraic fact that a tensor product of two rings of finite type is a ring of finite type.

This completes the proof. □

23. Seperated Morphisms

Separatedness is a fundamental property of morphisms of schemes. It serves as the analog of the Hausdorff condition in the context of manifolds. However, the standard topological definition of the Hausdorff condition is not directly applicable in abstract algebraic geometry, since the Zariski topology is never Hausdorff, and the underlying topological space of a scheme does not fully capture its geometric and algebraic structure. Grothendieck emphasized that one should define properties in terms of morphisms rather than objects. In this spirit, separatedness is defined as a property of morphisms of schemes, not of schemes themselves. liminary observations about diagonal morphisms.

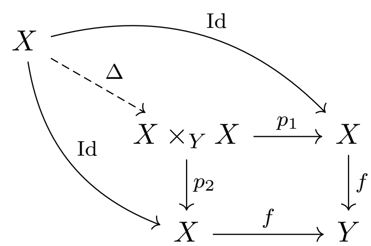
Definition 23.1. Let X, Y be schemes and $f : X \mapsto Y$ be a morphism of schemes.

(1) The **diagonal morphism** associated to a morphism $f : X \to Y$ is the unique morphism

$$\Delta : X \to X \times_Y X$$

whose composition with each of the projection morphisms $p_1, p_2 : X \times_Y X \to X$ is the identity morphism on X, i.e., $p_1 \circ \Delta = \mathrm{id}_X = p_2 \circ \Delta$.

(2) The morphism $f : X \to Y$ is said to be **separated** if the diagonal morphism Δ is a closed immersion.



Lemma 23.2. *Let X, Y be affine schemes. Then a morphism $f : X \to Y$ is separated.*

Proof. Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. Then R is naturally an S-algebra, and the fiber product $X \times_Y X$ is also affine, given by

$$X \times_Y X = \operatorname{Spec}(R \otimes_S R).$$

The diagonal morphism $\Delta : X \to X \times_Y X$ is induced by the multiplication map

$$R \otimes_S R \longrightarrow R,$$
$$r \otimes r' \mapsto rr'.$$

which is a surjective ring homomorphism. By Lemma 17.10, the diagonal morphism Δ is a closed immersion. □

Proposition 23.3. *Let X, Y affine schemes. A morphism $f : X \to Y$ is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.*

Proof. The forward direction is clear. Now assume that $\Delta(X) \subseteq X \times_Y X$ is a closed subset. Let $p_1 : X \times_Y X \to X$ denote the first projection. Since $p_1 \circ \Delta = \mathrm{Id}_X$, it follows that Δ is a homeomorphism onto its image $\Delta(X)$. Consider the induced morphism of sheaves:

$$\mathscr{O}_{X \times_Y X} \longrightarrow \Delta_* \mathscr{O}_X.$$

Let $x \in X$ and choose an open affine neighborhood $x \in U \subseteq X$ such that $f(U) \subset V$, where $V \subset Y$ is affine open. Then the fiber product $U \times_V U$ is an affine open neighborhood of $\Delta(x) \subseteq X \times_Y X$, and the restricted diagonal morphism

$$\Delta|_U : U \to U \times_V U$$

is a closed immersion. By Remark 17.11, the associated morphism of structure sheaves

$$\mathscr{O}_{U \times_V U} \to (\Delta|_U)_* \mathscr{O}_U$$

is surjective. Since this holds in a neighborhood of every point $x \in X$, it follows that the global morphism of sheaves

$$\mathscr{O}_{X \times_Y X} \to \Delta_* \mathscr{O}_X$$

is surjective. Hence, Δ is a closed immersion. □

Thanks Proposition 23.3 a morphism is separated if and only if the diagonal Δ is a closed subset of $X \times_Y X$—a purely topological condition on the diagonal. This is reminiscent of the definition of the Hausdorff condition in general topology. We now establish some basic properties of separated morphisms. Before doing so, we state a useful lemma that will be helpful in proving several of these properties.

Lemma 23.4. (Pullback/Fiber-Product Lemma) *Let $\mathscr{C}$ be a category admitting the following commutative diagram:*

$$\begin{array}{ccccc} F & \xrightarrow{f'} & E & \xrightarrow{g'} & D \\ \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

Suppose furthermore that the right square defines a fiber product in $\mathscr{C}$. Then the left square defines a fiber product in $\mathscr{C}$ if and only if the outer rectangle defines a fiber product in $\mathscr{C}$.

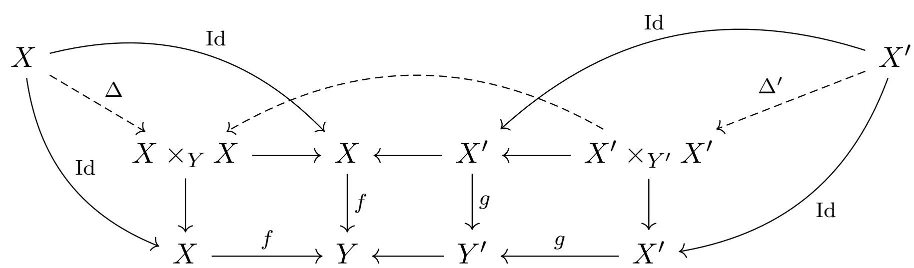
Proof. See [Awo10, Lemma 5.8]. □

Proposition 23.5. *The following is a is a list of properties of seperated morphisms.*

(1) Separated morphisms are stable under base extension.
(2) A composition of two separated morphisms is separated.
(3) A morphism $f : X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that the restriction $f^{-1}(V_i) \to V_i$ is separated for all i.
(4) Open and closed immersions are separated.

Proof. The proof is given below:

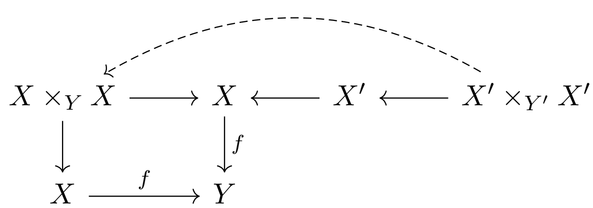
(1) Assume $f : X \to Y$ is separated. That is $\Delta : X \to X \times_Y X$ is a closed immersion. Let $g : X' \to Y'$ be obtained by a base change along $Y' \to Y$. Consider the diagram:



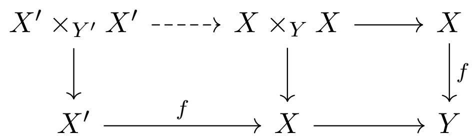
The middle and left squares define fiber products. By Lemma 23.4, the outer square determined by the middle and left squares defines a fiber product. Moreover, the map from $X' \times_{Y'} X'$ to $X \times_{Y'} X$ since the following square is trivially commutative:

$$\begin{array}{ccc} X & \longleftarrow & X' \times_Y' X' \\ \downarrow & & \downarrow \\ Y & \longleftarrow & X \end{array}$$

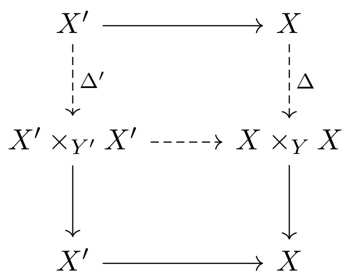
Therefore, the map from $X' \times_{Y'} X'$ to $X \times_{Y'} X$ is well-defined because of the universal property of fiber products. Therefore, we hone in one the following subdiagram:



That is, we have the following equivalent diagram:



We've already identified that the outer rectangle in the diagram above defines a fiber-product. But the right square in the diagram above also defines a fiber product. Therefore, Lemma 23.4 implies the left square in the diagram above also defines a fiber product. Thus, in the following diagram,



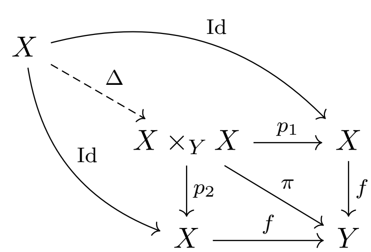
the outer rectangle and the bottom half are pullback diagrams, so the top half is a pullback square. Hence, $\Delta' : X' \to X' \times_{Y'} X'$ is indeed a closed immersion since closed immersions are stable under base extensions.

(2) This follows from a similar diagram chasing argument as in (1).

(3) Suppose V_i is an open cover of Y, and suppose $f^{-1}(V_i) \to V_i$ is separated for each i. We show that the image of the diagonal map

$$\Delta : X \to X \times_Y X$$

is closed. It suffices to show locally for some open cover $\mathcal{U}$ of $X \times_Y X$. If we let $\pi : X \times_Y X \to Y$ be the natural map, then $\pi^{-1}(V_i)$ gives an open cover of $X \times_Y X$. This implies that $\Delta^{-1}(\pi^{-1}(V_i)) \to \pi^{-1}(V_i)$ is closed for each i.



Conversely, (1) implies that separatedness is preserved by base change. Since the base change of $f : X \to Y$ by $V_i \hookrightarrow Y$ is just $f^{-1}(V_i) \to V_i$, the result immediately follows.

(4) The property of being an open/closed immersion is local on the target, and locally an open/closed immersion is a morphism of affine schemes. Since a morphism of affine schemes is seperated and the property of being seperated is local on the target, the claim follows.

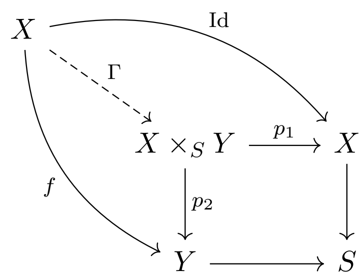
This completes the proof. □

We conclude this section with an application of the diagonal morphism by introducing the notion of the graph of a morphism.

Definition 23.6. Let X, Y be schemes over S. If $f : X \to Y$ is a morphism, then **graph** of f, denoted by Γ, is a morphism

$$\Gamma : X \to X \times_S Y$$

defined such that the following diagram commutes:



Proposition 23.7. (Hartshorne II.4.8) *Let X, Y be schemes over S and let Let $f : X \to Y$ be a morphism of schemes. If Y is separated, then the graph morphism, Γ, is a closed immersion. More precisely, it is obtained by the diagonal morphism over Y, $\Delta : Y \to Y \times_S Y$, by base extension.*

Proof. We first show that the graph of a morphism is obtained by a base change of the diagonal morphism $\Delta : Y \to Y \times_S Y$. That is, we show that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & X \times_S Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\times 1} \\ Y & \xrightarrow{\Delta} & Y \times_S Y \end{array}$$

is a fiber product diagram. In other words, we must verify that

$$X \cong Y \times_{Y \times_S Y} (X \times_S Y).$$

Given that $X \times_S Y$ is defined as a fiber product of schemes, the commutative diagram above yields a natural morphism

$$X \longrightarrow Y \times_{Y \times_S Y} (X \times_S Y).$$

It suffices to check that this morphism is an isomorphism locally on affine open subsets. Let $u : X \to S$ and $v : Y \to S$ be the structure morphisms, so that $u = v \circ f$. Choose affine open subsets $U = \operatorname{Spec} R \subset S$, $W = \operatorname{Spec} S \subset v^{-1}(U) \subset Y$, and $V = \operatorname{Spec} T \subset f^{-1}(W) \subset X$. The diagram then restricts to:

$$\begin{array}{ccc} V & \xrightarrow{\Gamma|_V} & V \times_U W \\ \downarrow{\scriptstyle f|_V} & & \downarrow{\scriptstyle f\times 1} \\ W & \xrightarrow{\Delta|_W} & W \times_U W \end{array}$$

Suppose the ring homomorphism $\varphi : S \to T$ is induced by f. Then, in the opposite category of rings (i.e., commutative rings), we obtain the following diagram:

$$\begin{array}{ccc} T & \xleftarrow[\gamma]{} & T \otimes_R S \\ {\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle \varphi\otimes 1} \\ S & \xleftarrow[\delta]{} & S \otimes_R S \end{array}$$

Here,

$$\gamma(r \otimes s) := r \cdot \varphi(s), \qquad \delta(s_1 \otimes s_2) := s_1 s_2.$$

It is straightforward to verify that this diagram commutes. To prove that the diagram is a pushout, we show that

$$T \cong S \otimes_{S \otimes_R S} (T \otimes_R S).$$

We define mutually inverse maps:

$$\phi : T \to S \otimes_{S \otimes_R S} (T \otimes_R S), \quad t \mapsto 1 \otimes (t \otimes 1),$$

$$\psi : S \otimes_{S \otimes_R S} (T \otimes_R S) \to T, \quad s_1 \otimes (t \otimes s_2) \mapsto \varphi(s_1 s_2) \cdot t.$$

One can check that both ϕ and ψ are well-defined and inverses of each other. Therefore, the diagram is indeed a pushout in the category of commutative rings, and hence a fiber product in the category of schemes. Since Δ is a closed immersion, it follows by base change that Γ is also a closed immersion. □

24. Proper Morphisms

Part 5. Appendix: Commutative Algebra

The purpose of this section is to review the definitions and results from commutative algebra. The algebra-geometry correspondence discussed in Section 2 is invoked from time to time to provide motivation for some concepts.

25. Prime and Maximal Ideals

Prime ideals and maximal ideals are the building blocks of algebraic geometry.

Definition 25.1. Let R be a ring.

(1) An ideal $\mathfrak{p}$ in R is prime if $\mathfrak{p} \neq R$ and for every $a, b \in R$, if $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
(2) An ideal $\mathfrak{m}$ in R is maximal if $\mathfrak{m} \neq R$ and there is no ideal I such that $\mathfrak{m} \subsetneq I \subseteq R$.

Remark 25.2. *Let R be an integral domain, and let $p \in R \backslash \{0\}$ not be a unit. By definition, the ideal (p) is prime if and only if for all $a, b \in R$ with $p \mid ab$, we have $p \mid a$ or $p \mid b$. That is, p is a prime element of R This is the origin of the name 'prime ideal.'*

Remark 25.3. *Let R be a ring. Every prime ideal of R in a ring a is a radical ideal. This is a straighforward consequence of the definition.*

Proposition 25.4. *Let I be an ideal in a ring R with $I \neq R$.*

(1) *I is a prime ideal if and only if R/I is an integral domain.*
(2) *I is a maximal ideal if and only if R/I is a field.*

Proof. The proof is given below:

(1) Passing to the quotient ring R/I, the definition of a prime ideal that I is prime if and only if for all $\overline{a}, \overline{b} \in R/I$ with $\overline{a}\overline{b} = 0$ we have $\overline{a} = 0$ or $\overline{b} = 0$ if and only if R/I is an integral domain.
(2) The definition of a maximal ideal (b) means exactly that the ring R/I has only the trivial ideals I/I and R/I which is equivalent to R/I being a field.

This completes the proof. □

Corollary 25.5. *Let R be a ring. Every maximal ideal of R in a ring a is prime ideal.*

Proof. This is because every field is an integral domain. □

A prime ideal is not necessarily a maximal ideal. However, this is the case in some situations.

Proposition 25.6. *Let R be a principal ideal domain (PID). Then every non-zero prime ideal is a maximal ideal.*

Proof. Let $\mathfrak{p}$ be a prime ideal. Since R is PID, we have $\mathfrak{p} = (p)$. If $\mathfrak{q} = (q)$ is such that $\mathfrak{p} \subseteq \mathfrak{q}$, then $p = qr$ for some $r \in R$. Hence, either $q \in \mathfrak{p}$ or $r \in \mathfrak{p}$. If $q \in \mathfrak{p}$, then $\mathfrak{q} \subseteq \mathfrak{p}$. If $r \in \mathfrak{p}$, then $r = pp'$ for some $p' \in R$ implies that

$$p = qr = qpp' \;\Rightarrow\; p(1 - qp') = 0$$

Since $p \neq 0$ and R is an integral domain, we have that $1 = qp'$. Hence, $\mathfrak{q} = R$. In any case, $\mathfrak{p}$ is a maximal ideal. □

Proposition 25.7. *(Atiyah-Macdonald 1.7) Let R be a ring in which every element x satisfies $x^n = x$ for some $n > 1$. Every prime ideal in R is a maximal ideal.*

Proof. Let $\mathfrak{p}$ be a prime ideal in A. Then $A/\mathfrak{p}$ is an integral domain. Let $\overline{x} \in A \setminus \mathfrak{p}$. By assumption, there is a $n > 1$ such that $x^n = x$. Passing to the quotient ring, we then have $\bar{x}^n = \bar{x}$. This is equivalent to $\bar{x}(\bar{x}^{n-1} - \bar{1}) = 0$. Since $A/\mathfrak{p}$ is an integral domain and $x \notin \mathfrak{p}$, we have $\bar{x}^{n-1} = \bar{1}$. If $n = 2$, then $\bar{x} = \bar{1}$ which is already a unit in $A \setminus \mathfrak{p}$. If $n > 2$, then $\bar{x}$ is a unit in $A \setminus \mathfrak{p}$ with inverse $\bar{x}^{n-2}$. Hence $A/\mathfrak{p}$ is a field, which implies $\mathfrak{p}$ is a maximal ideal. □

Proposition 25.8. *(Atiyah-Macdonald 1.11) A ring R is Boolean if $x^2 = x$ for all $x \in R$. In a Boolean ring R, we have:*

(1) $2x = 0$ *for all* $x \in R$*;*
(2) *Rvery prime ideal* $\mathfrak{p}$ *is maximal, and* $A/\mathfrak{p}$ *is a field with two elements;*
(3) *Every finitely generated ideal in* I *is a principal ideal.*

Proof. The proof is given below:

(1) In every Boolean ring, we have

$$2x = x + x = (x + x)^2 = 4x$$

Therefore, $2x = 0$.
(2) The argument in Proposition 25.7 shows that every prime ideal, $\mathfrak{p}$, is maximal. Then $A/\mathfrak{p}$ is a field. Since $A/\mathfrak{p}$ is an integral domain and $\bar{x}^2 = \bar{x}$ for every $\bar{x}$ in $A/\mathfrak{p}$, we have that $\bar{x} = \bar{0}, \bar{1}$. Therefore, $A/\mathfrak{p} \cong \mathbb{Z}_2$.
(3) Let $I = (a, b)$ and $J = (a + b + ab)$. Clearly, $J \subseteq I$. Note that

$$a(a + b + ab) = a^2 + ab + a^2b = a^2 + 2ab = a^2 = a$$

Hence $a \in J$. Similarly, $b \in J$. Therefore, $J \subseteq I$. The general claim follows by induction.

This completes the proof. □

Example 25.9. The following is a list of basic examples of prime ideals and maximal ideals.

(1) If p is a prime number, then $p\mathbb{Z} \subseteq \mathbb{Z}$ is a prime ideal. This can be checked by definition. In fact, $p\mathbb{Z}$ is a maximal ideal.
(2) If p is a prime number, $p\mathbb{Z}[x] \subseteq \mathbb{Z}[x]$ is a prime ideal. Indeed, $p\mathbb{Z}[x] = \ker \pi$, where

$$\pi : \mathbb{Z}[x] \to (\mathbb{Z}/p\mathbb{Z})[x], \qquad \pi\left(\sum a_i x^i\right) = \sum a_i x^i$$

Hence, $\mathbb{Z}[x]/p\mathbb{Z}[x] \cong (\mathbb{Z}/p\mathbb{Z})[x]$. The latter is an integral domain since $\mathbb{Z}/p\mathbb{Z}$ is a field.
(3) Let R be an integral domain. The ideal $(x) \subseteq R[x]$ is a prime ideal, since $R[x]/(x) \cong R$.
(4) Let $\mathbb{K}$ be a field. The ideal $(x) \subseteq \mathbb{K}[x]$ is a maximal ideal, since $\mathbb{K}[x]/(x) \cong \mathbb{K}$.

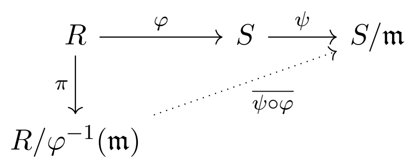
The following basic observation is quite important.

Proposition 25.10. *Let R, S be rings and let $\varphi : R \to S$ be a ring homomorphism.*

(1) *If* $\mathfrak{q} \subseteq S$ *is a prime ideal, then* $\varphi^{-1}(\mathfrak{q}) \subseteq R$ *is a prime ideal.*
(2) *If* $\mathfrak{q} \subseteq S$ *is a maximal ideal, then* $\varphi^{-1}(\mathfrak{q}) \subseteq R$ *is not necessarily a maximal ideal.*
(3) *If* φ *is surjective and* $\mathfrak{q} \subseteq S$ *is a maximal ideal, then* $\varphi^{-1}(\mathfrak{q}) \subseteq R$ *is a maximal ideal.*

Proof. The proof is given below:

(1) Note that $R/\varphi^{-1}(\mathfrak{q})$ is contained in the kernel of the map $R \to S/\mathfrak{q}$. Hence, have a map $R/\varphi^{-1}(\mathfrak{q}) \hookrightarrow S/\mathfrak{q}$. Since the latter is an integral domain, $R/\varphi^{-1}(\mathfrak{q})$ is also an integral domain. Hence, $\varphi^{-1}(\mathfrak{q})$ is a prime ideal.
(2) Let $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$. $\{0\}$ is a maximal ideal in $\mathbb{Q}$ but not in $\mathbb{Z}$ since $\mathbb{Z}$ is not a field.
(3) Consider the diagram:



Since $\psi \circ \varphi$ is surjective, $\pi \circ \overline{\psi \circ \varphi}$ is surjective. Thus $\overline{\psi \circ \varphi}$ is surjective. But $\ker \psi \circ \varphi = R/\varphi{-}1(\mathfrak{m})$. Hence, $\overline{\psi \circ \varphi}$ is injective. Hence.

$$R/\varphi^{-1}(m) \cong S/\mathfrak{m}$$

Since the latter is a field, $R/\varphi^{-1}(m)$ is also a field. Hence, $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal.

This completes the proof. □

A useful fact is that if $I \subseteq R$ is a proper ideal, then there is a maximal ideal $\mathfrak{m}$ in R with $I \subseteq \mathfrak{m}$. The proof uses Zorn's lemma.

Proposition 25.11. *Let R be a ring. Let I be an ideal in a ring R with $I \neq R$. Then I is contained in a maximal ideal, of R. In particular, every ring $R \neq 0$ has a maximal ideal.*

Proof. Let $\mathscr{M}$ be the set of all ideals J in R with $J \supseteq I$ and $J \neq R$. Let $\mathscr{A} \subset \mathscr{M}$ be a totally ordered subset, i.e., a family of proper ideals of R containing I such that for any two of these ideals, one is contained in the other. If $A = \emptyset$, then we can just take $I \in M$ as an upper bound for A. Otherwise, let

$$J' := \bigcup_{J \in \mathscr{A}} J$$

be the union of all ideals in A. We claim that this is an ideal:

(1) $0 \in J'$, since 0 is contained in each $J \in A$, and A is non-empty.
(2) If $a_1, a_2 \in J'$, then $a_1 \in J_1$ and $a_2 \in J_2$ for some $J_1, J_2 \in A$. But A is totally ordered, so without loss of generality, we can assume that $J_1 \subseteq J_2$. It follows that $a_1 + a_2 \in J_2 \subseteq J'$.
(3) If $a \in J'$, i.e., $a \in J$ for some $J \in A$, then $ra \in J \subseteq J'$ for any $r \in R$.

Moreover, J' certainly contains I, and we must have $J' \neq R$ since $1 \notin J$ for all $J \in A$, so that $1_R \notin J'$. Hence, $J' \in \mathscr{M}$, and it is certainly an upper bound for $\mathscr{A}$. By Zorn's Lemma, $\mathscr{M}$ has a maximal element, $\mathfrak{m}$, containing I. It can be easily checked that $\mathfrak{m}$ is a maximal ideal. If I is a proper ideal of R, then R/I contains a maximal ideal, $\mathfrak{m}'$. Then $\mathfrak{m} = \pi^{-1}(\mathfrak{m})$ is a maximal ideal of R[20] containing I. □

We end this section with some other applications of Zorn's lemma.

Proposition 25.12. *Let R be a ring.*

[20] π is surjective.

(1) *The nilradical of R, $\mathfrak{N}(R)$, is the intersection of all the prime ideals of R. That is,*

$$\mathfrak{N}(R) = \bigcap_{\substack{\mathfrak{p} \\ \mathfrak{p} \text{ is prime}}} \mathfrak{p}$$

(2) *Let I be an ideal of R. Then*

$$\sqrt{I} = \bigcap_{\substack{I \subseteq \mathfrak{p} \\ \mathfrak{p} \text{ is prime}}} \mathfrak{p}$$

(3) *The set of prime ideals of R has minimal elements with respect to inclusion.*

Proof. The proof is given below:

(1) Clearly, a nilpotent element belongs to every prime ideal. Hence the $\subseteq$ inclusion is true. Conversely, let $x \in R$ be a non-nilpotent element. In order to ptove the $\supseteq$ inclusion, it suffice to find a a prime ideal, $\mathfrak{q}$, not containing x. Let Σ be the set of all ideals $I \subseteq R$ such that $x^n \notin I$ for all $n \in \mathbb{N}$. Since x is not nilpotent, $\{0\} \in \Sigma \neq \emptyset$. If $\mathscr{C} = \{I_n\}_{n \in \mathbb{N}}$ is a chain in Σ, then $I = \bigcup_{n \in \mathbb{N}} I_n \in \Sigma$. Hence, Σ satisfies the assumption of Zorn's lemma. Let $\mathfrak{q} \in \Sigma$ be a maximal element. We show that $\mathfrak{q}$ is a prime ideal.

   If $y, z \notin \mathfrak{q}$, then $\mathfrak{q}$ is properly contained in $(y, \mathfrak{q})$ and $(z, \mathfrak{q})$, so these ideals are not in Σ. Hence, there exist integers $n, m \in \mathbb{N}$ and elements $a, c \in R$ and $b, d \in \mathfrak{q}$ such that $x^n = ay + b$ and $x^m = cz + d$. If $yz \in \mathfrak{q}$, we would get

$$x^{n+m} = acyz + (ayd + czb + bd) \in (yz, \mathfrak{q})$$

   which is a contradiction. Hence, $yz \notin \mathfrak{q}$. Hence, $\mathfrak{q}$ is a prime ideal.

(2) The proof is similar to (2).

(3) Let Σ be the set of all prime ideals with partial order Σ is given by reverse inclusion. $\Sigma \neq \emptyset$ since $\{0\} \in \Sigma$. Consider a chain of prime ideals in Σ:

$$\mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \cdots \mathfrak{p}_n \supseteq \cdots$$

   Let $\mathfrak{p} = \bigcap_{i \in \mathbb{N}} \mathfrak{p}_i$. Clearly, $\mathfrak{p}$ is an upper bound for the chain (under reverse inclusion). We show that $\mathfrak{p}$ is a prime ideal.

   Let $x, y \notin \mathfrak{p}$. Then $x \notin \mathfrak{p}_i$ and $y \notin \mathfrak{p}_j$ for some $i, j \geq 0$. WLOG, let $i < j$. We then have that both $x, y \notin \mathfrak{p}_j$. Since $\mathfrak{p}_j$ is prime, we must have $xy \notin \mathfrak{p}_j$. This in turn implies $xy \notin \mathfrak{p}$. Hence $\mathfrak{p}$ is a prime ideal. By Zorn's lemma, Σ has a has a maximal element, which is a minimal element in the set of all prime ideals given the reverse inclusion order.

This completes the proof. □

Proposition 25.13. *(Atiyah-Macdonald 1.9) Let R be a ring and let $I \subsetneq R$ be an ideal of R. Then $I = \sqrt{I}$ if and only if I is an intersection of prime ideals.*

Proof. The forward implication is clear since $\sqrt{I}$ is an intersection of prime ideals by Proposition 25.12(2). Conversely, let I be an intersection of prime ideals. Let $x \in \sqrt{I}$. Then there is a $n > 0$ such that $x^n \in I$. Therefore, x^n is contained in all prime ideals that define I. Hence, implies x is is contained in all prime ideals that define I. Hence, $x \in I$. □

Proposition 25.14. *(Atiyah-Macdonald 1.10) Let R be a ring, The following are equivalent:*

(1) *R has exactly one prime ideal.*
(2) *Every element of R is either a unit or nilpotent.*
(3) *$R/\mathfrak{N}(R)$ is a field.*

Proof. The proof is given below:

- (i) $\iff$ (ii): Assume R has exactly one prime ideal. Since every maximal ideal is a prime ideal, R has a unique maximal ideal. By Proposition 25.12(1) $\mathfrak{N}(R)$ is the intersection of all prime ideals of R. Hence, we must have that the nilradical is the unique prime ideal. If an element x is not nilpotent, it is not contained in the unique prime/maximal ideal. Therefore, x must be a unit; otherwise, x would be contained in the unique maximal ideal of R.
- (ii) $\iff$ (iii): Assume every element of R is either a unit or nilpotent. This clearly implies that $R/\mathfrak{N}(R)$ is a field.
- (iii) $\iff$ (i): Assume $R/\mathfrak{N}(R)$ is a field. Then $\mathfrak{N}(R)$ is a maximal ideal. Thus if $\mathfrak{p}$ is prime ideal in R, then $\mathfrak{p}$ should be the nilradical since the nilradical is a maximal ideal. Thus there is only one prime ideal.

This completes the proof. □

Proposition 25.15. *(Atiyah-Macdonald 1.14) Let R be a ring. Let Σ be the set of all ideals in which every element is a zero-divisor. Σ has maximal elements and every maximal element of $\mathcal{E}$ is a prime ideal. Hence, the set of zero-divisors in A is a union of prime ideals.*

Proof. A standard Zorn's Lemma type once again provides the correct argument. We omit the details. □

26. Localization

The purpose of this short section is to briefly review the details behind localization and to provide motivation for its use. We first appeal to Conjecture 2.8 to give a geometric motivation of the localization construction.

Remark 26.1. *(Geometric Motivation) Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set and let $R = A(X)$. Elements of $A(X)$ are equivalence classes of polynomial functions in $\mathbb{K}[x_1, \cdots, x_n]$. We continue the equivalence class of $f \in \mathbb{K}[x_1, \cdots, x_n]$ as f. Does it makes sense to consider fractions of such polynomial functions, i.e., rational functions*

$$X \to \mathbb{K}$$

$$x \mapsto \frac{f(x)}{g(x)}$$

for $f, g \in A(X)$. Of course, this does not work for global functions on all of X since g will in general have zeroes in X. But if we consider functions on "small enough" subsets of X, we can allow such fractions. This intuition is captured by the procedure of localization of rings. For example, let fixed point $a \in X$, and consider the set

$$S = \{f \in A(X) : f(a) \neq 0\}$$

be the set of all polynomial functions that do not vanish at a. It is easy to check that S is a multiplictaively closed set that does not contain 0. Hence $\mathfrak{p} = A(X) \setminus S$ is a prime ideal. Then the fractions $\frac{f}{g}$ for $f \in R$ and $g \in S$ can be thought of as rational functions that are well-defined "near" a. This construction amounts to analyzing the affine algebraic set, X, *locally around the point, a. by studying the corresponding ring of rational functions. We will see that localization of $A(X)$ at $\mathfrak{p}$ will make this intuition precise.*

Let us now introduce such "fractions" in a rigorous way. Let R be a ring, and $S \subseteq R$ the set of elements that we would like to become invertible. Note that this subset S of denominators for our fractions has to be closed under multiplication, for otherwise the formulas

$$\frac{a}{s} + \frac{a'}{s'} := \frac{as' + a's}{ss'}$$

and

$$\frac{a}{s} \cdot \frac{a'}{s'} := \frac{aa'}{ss'}$$

for addition and multiplication of fractions would not make sense. Moreover, we should have $1 \in S$ in order to make sure that fractions of the form $\frac{a}{1}$ for $a \in R$ make sense. Such sets are called multiplicatively closed subsets of R.

Proposition 26.2. *Let R be a ring and $S \subseteq R$ be a multiplicatively closed set. The relation*

$$(a, s) \sim (a', s') \iff \text{there is an element } u \in S \text{ such that } u(as' - a's) = 0$$

is an equivalence relation on $R \times S$. We denote the equivalence class of a pair $(a, s) \in R \times S$ by $\frac{a}{s}$. The set of all equivalence classes

$$S^{-1}R := \left\{ \frac{a}{s} : a \in R, s \in S \right\}$$

is called the localization of R at the multiplicatively closed set S. It is a ring together with the addition and multiplication

$$\frac{a}{s} + \frac{a'}{s'} := \frac{as' + a's}{ss'}, \qquad \frac{a}{s} \cdot \frac{a'}{s'} := \frac{aa'}{ss'}.$$

Proof. (Sketch) The proof is similar to the proof constructing $\mathbb{Q}$ from $\mathbb{Z}$. The only caveat is that we must account for the element $u \in S$ in the definition of the equivalence relation. Without the additional element $u \in S$, the relation $\sim$ would not be transitive. Therefore, we confirm transitivity. Assume $(a, s) \sim (a', s')$, $(a', s') \sim (a'', s'')$. Then there are $u, u' \in S$ such that

$$u(as' - a's) = 0, \quad u'(a's'' - a''s') = 0,$$

and it follows that

$$uu's's''(as'' - a''s) = uu'(as' - a's)s''^2 + uu'(a's'' - a''s')ss'' = 0.$$

Since S is multiplicatively closed, $uu's's'' \in S$. Hence, $(a, s) \sim (a'', s'')$, as required. The rest of proof is standard aripropetic.[21]. □

Example 26.3. The following is a list of some examples of the localization procedure:

(1) If $S = \{1\}$, then $S^{-1}R \cong R$.
(2) Let R be an integral domain and let S be the set of non-zero-divisors in R. Then S is multiplicatively closed and $S^{-1}R$ is the quotient field of R.

[21] If fact, if we set $u = u' = 1$ in the proof of transitivity, we could only show that $s's''(as'' - a''s) = 0$. Of course, this does not imply $as'' - a''s = 0$ if S contains zero-divisors. On the other hand, if S does not contain any zero-divisors, the condition $u(as' - a's) = 0$ for some $u \in S$ can obviously be simplified to $as' - a's = 0$.

(3) For a fixed element $a \in R$, let $S = \{a^n : n \in \mathbb{N}\}$. Then S is obviously multiplicatively closed. The corresponding localization $S^{-1}R$, often written as R_a, is called the localization of R at the element a.
(4) Let $\mathfrak{p}$ be a prime ideal of a ring R. Then $S = R \setminus \mathfrak{p}$ is multiplicatively closed. The resulting localization $S^{-1}R$ is usually denoted by $R_{\mathfrak{p}}$ and called the localization of R at the prime ideal $\mathfrak{p}$. In fact, this construction formalizes the intuition in Remark 26.1. Indeed, if $R = A(X)$ is the ring of functions on an affine algebraic set variety $X \subseteq \mathbb{A}^n$ and
$$\mathfrak{p} = \mathbb{I}(a) = \{f \in A(X) : f(a) = 0\},$$
the localization $A(X)_{\mathfrak{p}}$ is exactly the ring of "fractions near a" as in Remark 26.1.

We end this short section by mentioning some properties of localization.

Proposition 26.4. *Let R be a ring and let $S \subseteq R$ be a multiplicatively closed subset. Let $\varphi : R \to S^{-1}R$ be the ring morphism $a \mapsto a/1$.*

(1) *There is a one-to-one correspondence:*
$$\{\text{prime ideals in } S^{-1}R\} \longleftrightarrow \{\text{prime ideals } \mathfrak{q} \text{ in } R \text{ with } \mathfrak{q} \cap S = \emptyset\}$$
$$I \mapsto \langle \varphi(I) \rangle$$
$$\varphi^{-1}(I) \leftarrow I$$
(2) *If $S = R \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then there is a one-to-one correspondence:*
$$\{\text{prime ideals in } R_{\mathfrak{p}}\} \longleftrightarrow \{\text{prime ideals } \mathfrak{q} \text{ in } R \text{ with } \mathfrak{q} \subseteq \mathfrak{p}\}$$
$$I \mapsto \langle \varphi(I) \rangle$$
$$\varphi^{-1}(I) \leftarrow I$$
(3) *If $S = R \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then $R_{\mathfrak{p}}$ is a local ring. That is, $R_{\mathfrak{p}}$ has a unique maximal ideal given by*
$$\langle \varphi(\mathfrak{p}) \rangle = \left\{ \frac{a}{s} : a \in \mathfrak{p}, s \notin \mathfrak{p} \right\}$$

Proof. The proof is given below:

(1) The proof is skipped.
(2) This follows from (1).
(3) The formula for $\langle \varphi(\mathfrak{p}) \rangle$ is clearly justified. Every prime ideal in $R_{\mathfrak{p}}$ is contained in $\langle \varphi(\mathfrak{p})$. In particular, every maximal ideal in $R_{\mathfrak{p}}$ must be contained in $\langle \varphi(\mathfrak{p}) \rangle$. But this means that $\langle \varphi(\mathfrak{p}) \rangle$ is the only maximal ideal.

This completes the proof. □

27. Chain Conditions

We have worked with an arbitrary commutative rings with identity thus far. However, to derive more significant theorems, we must impose certain finiteness conditions. The most practical approach is through "chain conditions," which apply to both rings and modules. In what follows, let Σ be a set partially ordered by a relation $\sim$ (i.e., $\sim$ is reflexive and transitive, and is such that $x \sim y$ and $y \sim x$ together imply $x = y$).

Lemma 27.1. *Let $(\Sigma, \leq)$ be a partially ordered set. The following conditions are equivalent:*

(1) *Every non-empty subset $S \subseteq \Sigma$ contains a maximal element;*

(2) *Every sequence* $x_1 \leq x_2 \leq \cdots$ *in* Σ *is stationary. That is, there exists a* $n_0 \in \mathbb{N}$ *such that* $x_n = x_{n+1}$ *for all* $n \geq n_0$.

Similarly, the following conditions are equivalent:

3 *Every non-empty subset* $S \subseteq \Sigma$ *contains a minimal element;*

4 *Every sequence* $x_1 \geq x_2 \geq \cdots$ *in* Σ *is stationary. That is, there exists a* $n_0 \in \mathbb{N}$ *such that* $x_n = x_{n+1}$ *for all* $n \geq n_0$.

Proof. Suppose that every $\emptyset \neq S \subseteq \Sigma$ contains a maximal element, and let $x_1 \leq x_2 \leq \cdots$ be a sequence in Σ. If $S = \{x_n \mid \forall n \in \mathbb{N}\}$, we can find $n_0 \in \mathbb{N}$ such that x_{n_0} is a maximal element in S. Conversely, assume every sequence in Σ is stationary and suppose $\emptyset \neq S \subseteq \Sigma$ has no maximal element. Let $x_1 \in S$ and construct a sequence inductively: given $x_1 \leq \cdots \leq x_n$, let $S_n = \{x \in S \mid x > x_n\}$. This set is non-empty (otherwise $x_n \in S$ is maximal), so pick $x_{n+1} \in S_n$. This yields a non-stationary sequence. The equivalence of (3) and (4) is proved similarly. □

Definition 27.2. Let R be a ring and M an R-module. Let Σ be the set of all submodules of M.

(1) M is **Noetherian** if $(\Sigma, \subseteq)$ satisfies the equivalent conditions of Lemma 27.1. In other words, if every ascending chain

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

of submodules of M becomes stationary.

(2) M is **Artinian** if $(\Sigma, \supseteq)$ satisfies the equivalent conditions of Lemma 27.1. In other words, if every descending chain

$$M_1 \supseteq M_2 \supseteq M_3 \supset \cdots$$

of submodules of M becomes stationary

Remark 27.3. *The conditions of (1) and (2) in Lemma 27.1 are often referred to as the ascending and descending chain conditions, respectively.*

Remark 27.4. *We say that* R *is Noetherian/Artinian if* R *is a Noetherian/Artinian* R*-module. In other words,* R *is Noetherian/Artinian if and only if any ascending/descending chain of ideals becomes stationary.*

(1) Any field $\mathbb{K}$ is trivially Noetherian and Artinian as it has only the trivial ideals (0) and $\mathbb{K}$.

(2) A $\mathbb{K}$-vector space V is Noetherian and Artinian if and only if it is finite-dimensional.
   - If V is finite-dimensional, there can only be finite strictly ascending or descending chains of vector subspaces of V since the dimension has to be strictly increasing or decreasing in such a chain, respectively. Hence, a finite-dimensional $\mathbb{K}$-vector space V is both Noetherian and Artinian.
   - If V is infinite-dimensional, we can form a chain

     $$V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots$$

     with $\dim V_n = n$ for all $n \in \mathbb{N}$. Indeed, simply set $V_0 = 0$, and $V_{n+1} = V_n + \langle v_{n+1} \rangle$ with $v_{n+1} \notin V_n$ for all $n \in \mathbb{N}$. Clearly, this chain is not stationary. Hence, an infinite-dimensional $\mathbb{K}$-vector space V is not Noetherian.
     Similarly, we can also find an infinite descending chain

     $$V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots$$

of infinite-dimensional subspaces of V with $\dim(V/V_n) = n$ for all n. Indeed, let $V_0 = V$, and $V_{n+1} = V_n \cap \ker \varphi_{n+1}$ for some linear map $\varphi_{n+1} : V \to k$ that is not identically zero on V_n. Then

$$V/V_n \cong (V/V_{n+1})/(V_n/V_{n+1})$$

and so $\dim(V_n/V_{n+1}) = 1$ implies $\dim(V/V_n) = n$ for all n by induction. Hence, an infinite-dimensional $\mathbb{K}$-vector space V is not Artinian.

(3) $\mathbb{Z}$ is Noetherian. Indeed, if we had a strictly increasing chain of ideals

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

in $\mathbb{Z}$, then certainly $I_2 \neq 0$, and thus $I_2 = n\mathbb{Z}$ for some non-zero $n \in \mathbb{Z}$. But there are only finitely many ideals in $\mathbb{Z}$ that contain I_2 since they correspond to ideals of the finite ring $\mathbb{Z}/n\mathbb{Z}$. Hence, the chain must be stationary.

On the other hand, $\mathbb{Z}$ is not Artinian, since there is an infinite decreasing chain of ideals

$$\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supset 8\mathbb{Z} \supsetneq \cdots .$$

(4) Let $R = R[x_1, x_2, \cdots, x_n, \cdots,]$ be the polynomial ring over R in infinitely many variables. Then R is neither Noetherian nor Artinian, since there are infinite chains of ideals

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subset \cdots$$

and

$$(x_1) \supsetneq (x_1^2) \supsetneq (x_1^3) \supsetneq \cdots .$$

We have another equivlanet characterization of Noetherian modules.

Proposition 27.5. *Let M be a R-module. $M$is Noetherian if and only if every submodule of M is finitely generated.*

Proof. Assume that we have a submodule $N \subseteq M$ that is not finitely generated. Then we can recursively pick $m_1 \in N$ and $m_{n+1} \in N \setminus \langle m_1, \ldots, m_n \rangle$ for $n \in \mathbb{N}$, and obtain a chain

$$M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$$

in M. This is a contradiction since M is Noetherian.

Let

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

be a chain of submodules of M. Then $N = \bigcup_{n\in\mathbb{N}} M_n$ is also a submodule of M. By assumption, N can be generated by finitely many elements $m_1, \ldots, m_r \in N$. We must have $m_i \in M_{n_i}$ for all $i = 1, \ldots, r$ and some $n_1, \ldots, n_r \in \mathbb{N}$. With $n = \max\{n_1, \ldots, n_r\}$, we then have $m_1, \ldots, m_r \in M_n$. Hence,

$$N = \langle m_1, \ldots, m_r \rangle \subseteq M_n \subseteq N,$$

which implies $M_k = M_n = N$ for all $k \geq n$. □

The one central result on Noetherian rings is Hilbert's Basis Theorem.

Proposition 27.6. ***(Hilbert's Basis Theorem)*** *If R is a Noetherian ring, $R[x]$ is also a Noetherian ring.*

PROOF. Assume that $R[x]$ is not Noetherian. Then there is an ideal $I \subseteq R[x]$ that is not finitely generated. We can therefore pick elements $f_0, f_1, f_2, \ldots \in I$ as follows: let $f_0 \in I$ be a non-zero polynomial of minimal degree, and for $k \in \mathbb{N}$, let f_{k+1} be a polynomial of minimal degree in $I \setminus \langle f_0, \ldots, f_k \rangle$. For all $k \in \mathbb{N}$, let $d_k \in \mathbb{N}$ be the degree and $a_k \in R$ the leading coefficient of f_k, so that we can write

$$f_k = a_k x^{d_k} + (\text{lower order terms}).$$

Note that $d_k \leq d_{k+1}$ for all k by construction of the polynomials. Since R is Noetherian, the chain of ideals

$$(a_0) \subseteq (a_0, a_1) \subseteq (a_0, a_1, a_2) \subseteq \cdots$$

becomes stationary. Hence, we we must have

$$a_{n+1} = c_0 a_0 + \cdots + c_n a_n$$

for some $n \in \mathbb{N}$ and $c_0, \ldots, c_n \in R$. We can therefore cancel the leading term in f_{n+1} by subtracting a suitable linear combination of $f_0, \ldots, f_n$:

$$f'_{n+1} := f_{n+1} - \sum_{k=0}^{n} c_k x^{d_{n+1} - d_k} f_k.$$

In this polynomial f'_{n+1}, the $x^{d_{n+1}}$-coefficient is $a_{n+1} - c_0 a_0 - \cdots - c_n a_n = 0$. But this means that $\deg f'_{n+1} < \deg f_{n+1}$. Hence, $f'_{n+1} \in \langle f_0, \ldots, f_n \rangle$. As $f_{n+1} \notin \langle f_0, \ldots, f_n \rangle$, we must have $f'_{n+1} \notin \langle f_0, \ldots, f_n \rangle$ as well. This is a contradiction. Hence, $R[x]$ is Noetherian. ☐

Remark 27.7. *An inductive argument implies that if R is a Noetherian ring, $R[x_1, \cdots, x_n]$ is a Noetherian ring for any $n \geq 1$.*

28. Nakayama's Lemma

The technique used in proving Lemma 29.4 can be used to prove result that is quite important in algebraic geometry, Nakayama's lemma. Nakayama's lemma is the following statement:

Proposition 28.1. *Suppose R is a ring, I is an ideal of R, and M is a finitely generated R-module, such that $M = IM$. Then there exists an $a \in R$ with $a - 1 \in I$ such that $aM = 0$.*

Here is the argument that proves Nakayama's lemma.

PROOF. Let M is generated by $m_1, \ldots, m_n$. Since $M = IM$, we have $m_i = \sum_j a_{ij} m_j$ for some $a_{ij} \in I$. Thus,

$$(\mathrm{Id}_{n \times n} - A)\vec{m} = 0$$

where $A = (a_{ij})$. Multiplying both sides of the equation on the left by $\mathrm{adj}(a\mathrm{Id}_{n \times n} - A)$, we obtain

$$\det(\mathrm{Id}_{n \times n} - A) m_i = 0$$

for each $1 \leq i \leq n$. Hence, $\det(\mathrm{Id}_{n \times n} - A) = 0$. When we expand $\det(\mathrm{Id}_{n \times n} - A)$, as A has entries in I, we get an expression of the form $1 + b$ for some $b \in I$. Let $a = 1 + b$. Hence, a annihilates M and we have $a - 1 \in I$. ☐

Corollary 28.2. *The following statements are true:*

(1) *Let M be a finitely generated R-module and let I be an ideal such that I is contained in every maximal ideal of R. If $IM = M$, then $M = 0$.*

(2) *Let M be a finitely generated R-module and let I be an ideal such that I is contained in every maximal ideal of R. If $N \subseteq M$ is a submodule satisfying $M = N + IM$ (or $N/IN \to M/IM$ is surjective), we must have $M = N$.*

(3) *Let R be a local ring with unique maximal ideal $m \subseteq R$. Then, for any finitely generated R-module M, the quotient $M/\mathfrak{m}M$ is canonically a vector space over the field $R/\mathfrak{m}$. Furthermore, if $x_1, \ldots, x_n \in M$ are elements whose residue classes $\overline{x_1}, \ldots, \overline{x_n} \in M/\mathfrak{m}M$ generate this vector space, then $M = \sum_{i=1}^{n} Rx_i$.*

Proof. The proof is given below:

(1) By Nakayama's lemma, there exists some $a \in R$ such that $a - 1 \in I$ and $aM = 0$. $a \in R$ is invertible. Indeed, if this is not the case, then $(a) \subsetneq R$ and a is contained in some maximal, $\mathfrak{m}$, ideal of R. Since $a - 1 \in \mathfrak{m}$, we have, $1 \in \mathfrak{m}$, a contradiction. Since a is invertible, we must have that $M = 0$.

(2) Note that $M = N + IM$ implies $M/N = I(M/N)$. Since M is finitely generated, M/N, 'is finitely generated, (1) implies $M/N = 0$. Hence, $M = N$.

(3) Clearly, $M/\mathfrak{m}M$ is a vector space over $R/\mathfrak{m}$. From $M/\mathfrak{m}M = \sum_{i=1}^{n}(R/\mathfrak{m})\,\overline{x_i}$ we conclude

$$M = \sum_{i=1}^{n} Rx_i + \mathfrak{m}M$$

(2) implies $M = \sum_{i=1}^{n} Rx_i$.

□

Remark 28.3. *Nakayama's lemma is not necessarily true if M is not a finitely generated R-module. Indeed, let $M = \mathbb{Q}$ thought of as a $\mathbb{Z}$-module (abelian group). Let $I = (2)$. If Nakayama's lemma were true, we would have an $m \in \mathbb{Z}$ such that $m\mathbb{Q} = 0$ and $m - 1 \in I$. The latter condition implies that m is an odd number. Clearly, Then $m\mathbb{Q} \neq 0$, a contradiction.*

29. Integral Extensions

Recall that if $\mathbb{K} \subseteq \mathbb{K}'$ is an algebraic extension of fields, then for any $\alpha \in \mathbb{K}'$, there is a *monic* polynomial, f, with coefficients in $\mathbb{K}$ such that $f(\alpha) = 0$. This implies $\mathbb{K}(\alpha)$ is a finite algebraic extension. This means that the field extension $\mathbb{K} \subseteq \mathbb{K}(\alpha)$ is quite easy to deal with, since we can use the whole machinery of finite-dimensional linear algebra.

What happens now if instead of an extension $\mathbb{K} \subseteq \mathbb{K}'$ of *fields* we consider an extension $R \subseteq R'$ of *rings*? We can certainly still have a look at elements $a \in R'$ satisfying a polynomial relation

$$c_n a^n + c_{n-1} a^{n-1} + \ldots + c_0 = 0$$

with $c_0, \ldots, c_n \in R$ (and not all of them being 0). But now it will in general not be possible to divide this equation by its leading coefficient c_n to obtain a monic relation. Consequently, we can in general not use this relation to express higher powers of a in terms of lower ones, and hence the R-algebra $R[a]$ generated by a over R need not be a finite R-module. A simple example for this can already be found in the ring extension $\mathbb{Z} \subseteq \mathbb{Q}$: for example, the rational number $a = \frac{1}{2}$ satisfies a (non-monic) polynomial relation $2a - 1 = 0$ with coefficients in $\mathbb{Z}$, but certainly $\mathbb{Z}[a]$, which is the ring of all rational numbers with a finite binary expansion, is not a finitely generated $\mathbb{Z}$-module. This motivates the definition:

Definition 29.1. Let $R \subseteq R'$ be rings. An element a of R' of R is **integral** over R if there is a *monic* polynomial $f \in R[x]$ with $f(a) = 0$, i.e., if there are $n \in \mathbb{N}_{>0}$ and $c_0, \dots, c_{n-1} \in R$ with $a^n + c_{n-1}a^{n-1} + \dots + c_0 = 0$. We say that $R \subseteq R'$ is an **integral extension** if every element of R' is integral over R.

Remark 29.2. *More generally, suppose $\varphi : R \to R'$ is a ring morphism. We say $a \in R'$ is integral over R if a satisfies some monic polynomial where the coefficients lie in $\varphi(R)$. A ring morphism $\varphi : R \to R'$ is an* ***integral morphism*** *if every element of R' is integral over $\varphi(R)$.*

Example 29.3. Let R be a unique factorization domain, and let R' be its quotient field. Then $a \in R'$ is integral over R if and only if $a \in R$[22].

The following lemma provides a useful trick to check for integrality.

Lemma 29.4. *Let $R \subseteq R'$ be a ring extension. Then $a \in R'$ is integral over R if and only if it is contained in a subalgebra of R' that is a finitely generated R-module.*

Proof. If a satisfies a monic polynomial equation of degree n, then the R-submodule of R' generated by $1, a, \dots, a^{n-1}$ is closed under multiplication, and hence a subalgebra of R'. Conversely, assume that a is contained in a subalgebra R'' of R' that is a finitely generated R-module. Choose a finite generating set $m_1, \dots, m_n$ of R'' (as an R-module). Then $am_i = \sum_j b_{ij}m_j$, for some $b_{ij} \in R$. That is,

$$(a\mathrm{Id}_{n\times n} - B)\vec{m} = \vec{0}.$$

Here B is the coefficient matrix of the coefficients b_{ij}'s and $\vec{m}$ is a column vector for the variables m_i's. Recall that a matrix M has an adjugate matrix $\mathrm{adj}(M)$ such that $\mathrm{adj}(M)M = \det(M)\mathrm{Id}_n$. Multiplying by $\mathrm{adj}(a\mathrm{Id}_{n\times n} - B)$, we get:

$$\det(a\mathrm{Id}_{n\times n} - B)m_i = 0 \qquad 1 \le i \le n$$

Since $\det(a\mathrm{Id}_{n\times n} - B)$ annihilates the generating elements m_i, and hence every element of R'', including 1. Hence, $\det(a\mathrm{Id}_{n\times n} - B) = 0$. But expanding the determinant yields an integral equation for a with coefficients in R. □

The above result has some easy corollaries:

Corollary 29.5. *Let $R \subseteq R'$ be a ring extension. The elements of R' integral over R form a subring of R'. If R' is a finite ring extension in the sense that R' is a finitely generated R-module, then $R \subseteq R'$ is an integral extension.*

Proof. If $a, b \in R'$ are integral over R, then $a \in R[x_1, \cdots, x_n]$ and $b \in R[y_1, \cdots, y_m]$ for $x_1, \cdots, x_n$ in R' and $y_1, \cdots, y_m$ in R' . Then

$$a \pm b,\ ab \in R[x_1, \cdots, x_n, y_1, \cdots, y_m]$$

Hence, integral elements form a subring. The other statement is clear since if R' is a finite R-algebra, then $R' = R[x_1, \cdots, x_n]$ as a finitely generating R-module. □

Corollary 29.6. ***(Transitivity)*** *Suppose $R \subseteq R'$ and $R' \subseteq R'''$ are integral ring extensions. Then $R \subseteq R''$ is an integral extension.*

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[22]The proof is similar to the argument of the rational roots theorem, so we omit details.

Proof. Let $a \in R''$. As a is integral over R', there are $n \in \mathbb{N}_{>0}$ and elements $c_0, \ldots, c_{n-1}$ of R' such that

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$$

Then a is also integral over $R[c_0, \ldots, c_{n-1}]$. In addition, we know that $c_0, \ldots, c_{n-1}$ are integral over R. Hence, $R[c_0, \ldots, c_{n-1}, a]$ is finite over $R[c_0, \ldots, c_{n-1}]$ and $R[c_0, \ldots, c_{n-1}]$ is finite over R. Therefore $R[c_0, \ldots, c_{n-1}, a]$ is finite over R[23], and thus a is integral over R. □

Below we catalog various algebraic properties of integral extensions:

Proposition 29.7. *Let $R \subseteq R'$ be an integral extension. The following are true:*

(1) *If I is an ideal of R', then R'/I is an integral extension ring of $R/(I \cap R)$. More generally,*
(2) *If S is a multiplicative, closed subset of R, then $S^{-1}R'$ is an integral extension ring of $S^{-1}R$[24].*
(3) *$R'[x]$ is an integral extension ring of $R[x]$.*

Proof. The proof proceeds as follows:

(1) Note that the natural map

$$R/(I \cap R) \to R'/I \qquad \overline{a} \mapsto \overline{a}$$

is well-defined and injective. Hence, we can regard R'/I as an extension ring of $R/(I \cap R)$. Moreover, for all $a \in R'$, there is a monic relation $a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$ with $c_0, \ldots, c_{n-1} \in R$, and hence by passing to the quotient also $\overline{a}^n + \overline{c}_{n-1}\overline{a}^{n-1} + \cdots + \overline{c}_0 = 0$. So $\overline{a}$ is integral over $R/(I \cap R)$.

(2) The ring morphism $S^{-1}R \to S^{-1}R'$, $a/s \mapsto a/s$ is obviously well-defined and injective. Moreover, for $a/s \in S^{-1}R'$, we have a monic relation $a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$ with $c_0, \ldots, c_{n-1} \in R$, and thus also

$$\left(\frac{a}{s}\right)^n + \frac{c_{n-1}}{s}\left(\frac{a}{s}\right)^{n-1} + \cdots + \frac{c_0}{s^n} = 0.$$

Hence a/s is integral over $S^{-1}R$.

(3) Let $f = a_n x^n + \cdots + a_0 \in R'[x]$, i.e., $a_0, \ldots, a_n \in R'$. Then $a_0, \ldots, a_n$ are integral over R, so also over $R[x]$, and thus $R[x][a_0, \ldots, a_n] = R[a_0, \ldots, a_n][x]$ is integral over $R[x]$ by Lemma 29.4. In particular, this means that f is integral over $R[x]$.

This completes the proof. □

Corollary 29.8. *Let $\varphi : R \to R'$ be an integral morphism.*

(1) *If $I \subseteq R$ and $I \subseteq R'$ are ideals satisfying $\varphi(I) \subseteq I'$, then the homomorphism $R/I \to R'/I'$ induced by φ is an integral morphism.*
(2) *For any multiplicative system $S \subseteq R$, the induced homomorphism $R_S \to R'_{\varphi(S)}$ is an integral morphism.*

Ring extensions arise naturally in affine algebraic geometry:

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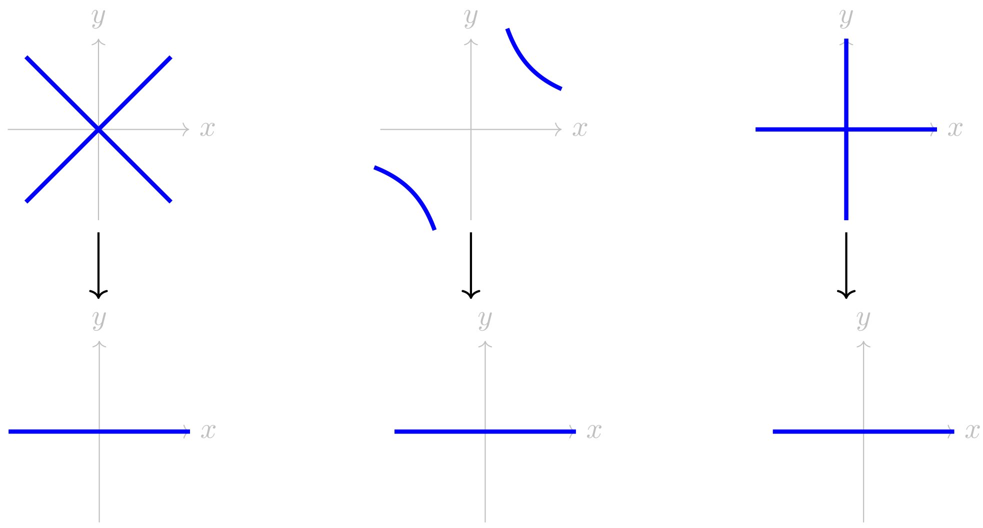
[23] Here we use that if $R \subset R'$ and $R' \subset R''$ are finite, then so is $R \subset R''$.

[24] Localization is not covered in these notes.



$f = y^2 - x^2$ $\qquad$ $f = xy - 1$ $\qquad$ $f = xy$

Example 29.9. Let $k = \mathbb{C}$. Consider $X = \mathbb{V}(f) \subseteq \mathbb{A}^2$ and $Y = \mathbb{A}^1$ for some $f \in \mathbb{C}[x, y]$. If $\pi : X \to Y$, $(x, y) \mapsto x$ is the cannonical projection morphism, then induced morphisms on coordiante rings is a ring extension:

$$A(Y) = \mathbb{C}[x] \subseteq A(X) = \mathbb{C}[x, y]/(f).$$

Some examples of such morphisms are sketched on the next page in Example 29.9. Note that we only plot real points to keep the pictures $\mathbb{R}^1$ and $\mathbb{R}^2$. Which of these ring extensions are integral extensions? If $f(x, y) = y^2 - x^2$. We then have a ring extension

$$\mathbb{C}[x] \subseteq \mathbb{C}[x, y]/(y^2 - x^2)$$

The ring extension is generated by $\overline{x}$ and $\overline{y}$. Clearly, $\overline{x}$ is integral over $\mathbb{C}[x]$ since x is integral over $\mathbb{C}[x]$. Moreover, $\overline{y}$ is integral over $\mathbb{C}[x]$ since it satisfies the polynomial relation $\overline{y}^2 - \overline{x}^2 = 0$.

Remark 29.10. *If $f(x, y) = xy$ or $f(x, y) = xy - 1$, then it turns out that the ring extensions $\mathbb{C}[x, y]/(f)$ are not integral ring extensions. How does one prove this claim? We shall return to this question later in this section.*

Corollary 29.5 implies that integral elements of a ring extension always form a ring themselves. This leads to the notion of integral closure.

Definition 29.11. Let $R \subseteq R'$ be a ring extension.

(1) The set $\overline{R}$ of all integral elements in R' over R is a ring with $R \subseteq \overline{R} \subseteq R'$. It is called the **integral closure** of R in R'. We say that R is **integrally closed** in R' if $\overline{R} = R$.

(2) An integral domain R is called integrally closed or **normal** if it is integrally closed in its quotient field $\mathrm{Quot}(R)$.

Example 29.12. Every unique factorization domain R is normal, since by Example 29.3 the only elements of $\mathrm{Quot}(R)$ that are integral over R are the ones in R.

Example 29.13. Let $R = A(X)$ be the coordinate ring of an affine algebraic set X.

(1) Let $R = \mathbb{C}[x]$, corresponding to $X = \mathbb{A}^1$. Since R is a UFD, we know that R is normal. In fact, this can also be understood geometrically: the only way a rational function ϕ on $\mathbb{A}^1$ can be ill-defined at a point $a \in \mathbb{A}^1$ is that it has a pole, i.e., that it is of the form $x \mapsto \frac{f}{(x-a)^k}$ for some $k \in \mathbb{N}_{>0}$ and $f \in \operatorname{Quot} R$ that is well-defined and non-zero at a. But then ϕ cannot satisfy a monic relation of the form

$$\phi^n + c_{n-1}\phi^{n-1} + \cdots + c_0 = 0$$

with $c_0, \dots, c_{n-1} \in \mathbb{C}[x]$ since ϕ^n has a pole of order kn at a which cannot be canceled by the lower order poles of the other terms $c_{n-1}\phi^{n-1} + \cdots + c_0 = 0$. Hence any rational function satisfying such a monic relation is already a polynomial function.

Consider the context of Example 29.9 above. Recalling that the inverse image of a prime ideal under any ring morphism is a prime ideal, then the ring extension $A(Y) \subseteq A(X)$ is such that for any prime ideal $\mathfrak{p}$ in $A(X)$, its inverse image under the ring extension, which is just $A(Y) \cap \mathfrak{p}$, is a prime ideal. Invoking Remark 2.19, geometrically this means that the image of an irreducible affine algebraic subset in X under the projection map, π, is an irreducible affine algebraic subset of Y. When is the converse true? That is, when is any prime ideal $\mathfrak{q}$ of $A(Y)$ of the form $\mathfrak{q} = \mathfrak{p} \cap A(Y)$, where $\mathfrak{p}$ is a prime ideal of $A(X)$? Geometrically, when is an irreducible algebraic subset of Y the image under the projection map, π, of an irreducible algebraic subset of X? It turns out such a result holds when the ring extension under investigation is an integral ring extension.

Lemma 29.14. ***(Lying Over Theorem)*** *Let $\varphi : R \to R'$ be an injective integral ring extension and let $\mathfrak{p}$ be a prime ideal in R. Then the following are true:*

(1) *Suppose $R \subseteq R'$. If R' is a field, then R is a field.*
(2) *Suppose $R \subseteq R'$ such that R and R' are integral domains. If R is a field, then R' is a field.*
(3) *Suppose $R \subseteq R'$. Let $\mathfrak{q}'$ be a prime ideal of R' and let $\mathfrak{q} = \mathfrak{q} \cap R$. Then $\mathfrak{q}$ is a maximal ideal if and only if $\mathfrak{q}'$ is a maximal ideal.*
(4) *There is a prime ideal $\mathfrak{p}'$ of R' with $\varphi^{-1}(\mathfrak{p}') = \mathfrak{p}$*

Remark 29.15. *In the context of the study of affine schemes, we can interpret the lying over theorem as follows: if $\varphi : R \to R'$ is an injective integral extension, then Spec $R' \to$ Spec R is surjective.*

Proof. (Lemma 29.14) The proof proceeds as follows:

(1) Suppose $R \subseteq R'$ and that R' is a field. Let $a \in R \setminus \{0\}$. Let $b = 1/a \in R'$. Since b is integral over R, we can write

$$b^r + \alpha_1 b^{r-1} + \ldots + \alpha_r = 0$$

for some positive integer r and some $\alpha_1, \ldots, \alpha_r \in R$. Hence,

$$1 + \alpha_1 a + \ldots + \alpha_r a^r = 0$$

Therefore,

$$\frac{1}{a} = -\alpha_1 - \alpha_2 a - \ldots - \alpha_r a^{r-1} \in R$$

We conclude that a is invertible in R. Since this holds for every nonzero $a \in R$, it follows that R is a field.

(2) Suppose R and R' are integral domains and R is a field. Let $b \in R'$, $y \neq 0$. Let

$$b^r + \alpha_1 b^{r-1} + \ldots + \alpha_r = 0 \quad (\alpha_i \in R)$$

such that r is the smallest possible positive integer for which the above equation is true. Since R' is an integral domain, $\alpha_r \neq 0$. Hence

$$b^{-1} = -\frac{1}{\alpha_r}(b^{r-1} + \alpha_1 b^{r-2} + \ldots + a_{r-1}) \in R'.$$

Hence R' is a field.

(3) By Proposition 29.7, $R'/\mathfrak{q}'$ is integral over $R/\mathfrak{q}$, and both these rings are integral domains. The result follows by (2).

(4) Since φ is injective, R can be thought of as a subring of R'. WLOG, we make this identification below. Let $S = R \setminus \mathfrak{p}$ be a multiplicative closed subset of R. Clearly, S is also a multiplicative closed subset of R'. Hence, we can localize both R and R' at $\mathfrak{p}$. By Proposition 29.7, $R_\mathfrak{p} \subseteq R'_\mathfrak{p}$ is an integral extension and the diagram

$$\begin{array}{ccc} R & \hookrightarrow & R' \\ \downarrow{\scriptstyle \alpha} & & {\scriptstyle \beta}\downarrow \\ R_\mathfrak{p} & \hookrightarrow & R'_\mathfrak{p} \end{array}$$

commutes. Let $\mathfrak{n}$ be the unique maximal ideal of $R'_\mathfrak{p}$. Then $\mathfrak{m} = \mathfrak{n} \cap R_\mathfrak{p}$ of $R_\mathfrak{p}$ by (3). Since $R_\mathfrak{p}$ is a local ring, $\mathfrak{m}$ is the unique maximal ideal of $R_\mathfrak{p}$. If $\mathfrak{p}' = \beta^{-1}(\mathfrak{n})$, then $\mathfrak{p}'$ is a prime ideal of R', and we have $\mathfrak{p}' \cap R = \alpha^{-1}(\mathfrak{m}) = \mathfrak{p}$.

This completes the proof. □

Remark 29.16. *Assume that an integral ring extension $A(Y) \to A(X)$ corresponds to a polynomial morphism $X \to Y$. of affine algebraic sets. Lemma 29.14(c) is essentially a statement of a statement about the property of fibers of the morphism $X \to Y$: it says that in integral ring extensions only maximal ideals can contract to maximal ideals, i.e., that points in X are the only irreducible algebraic sets that can map to a single point in Y.*

Example 29.17. Let's go back to Example 29.9. Consider the extension $R = \mathbb{C}[x] \subseteq R' = \mathbb{C}[x,y]/(f)$ where $f(x,y) = xy - 1$. The prime ideal (x) in $\mathbb{C}[x]$ does not lie over any prime ideal of $\mathbb{C}[x,y]/(f)$. If not, then there is a prime ideal ı of $\mathbb{C}[x,y]$ such that

$$(x) = \mathbb{C}[x] \cap \mathfrak{p} \qquad (xy-1) \subseteq \mathfrak{p}$$

In particular, both $xy - 1, xy \in \mathfrak{p}$. Since $-1 = xy - 1 - xy$, -1 and hence $+1$ must be in $\mathfrak{p}$, a contradiction. Hence, the extension $R = \mathbb{C}[x] \subseteq R' = \mathbb{C}[x,y]/(f)$ is not an integral extension.

It is still not clear how to prove the extension $\mathbb{C}[x] \subseteq \mathbb{C}[x,y]/(f)$ is not an integral extension when $f(x,y) = xy$. We need another property of integral extensions to argue that this is indeed the case.

Proposition 29.18. ***(Incomparability)*** *Let $R \subseteq R'$ be an integral ring extension. If $\mathfrak{p}'$ and $\mathfrak{p}'$ are distinct prime ideals in R' with $\mathfrak{p}' \cap R = \mathfrak{p}' \cap R$ then $\mathfrak{p}' \not\subseteq \mathfrak{q}'$ and $\mathfrak{q}' \not\subseteq \mathfrak{p}'$.*

Proof. Assume $\mathfrak{p}' \subseteq \mathfrak{q}'$. We show that $\mathfrak{q}' \subseteq \mathfrak{p}$, yielding a contradiction. Assume there is an element $a \in \mathfrak{q}' \setminus \mathfrak{p}'$. By Proposition 29.7, $R'/\mathfrak{p}'$ is integral over $R/(\mathfrak{p}' \cap R)$, so there is a monic relation

$$\overline{a}^n + \overline{c_{n-1}}\overline{a}^{n-1} + \cdots + \overline{c_0} = 0$$

in $R'/\mathfrak{p}'$ with $c_0, \ldots, c_{n-1} \in R$. Pick such a relation of minimal degree n. Since $a \in \mathfrak{q}'$, this relation implies $\overline{c}_0 \in \mathfrak{q}'/\mathfrak{p}'$, but as $c_0 \in R$ too we conclude that

$$\overline{c_0} \in (\mathfrak{q}' \cap R)/(\mathfrak{p}' \cap R) = (\mathfrak{q}' \cap R)/(\mathfrak{q}' \cap R) = 0.$$

Hence, the monic relation has no constant term. But since $a \neq 0$ in the integral domain $R'/\mathfrak{p}'$, we can then divide the relation by a to get a monic relation of smaller degree, a contradiction. □

Example 29.19. Let's go back to Example 29.9. Consider the extension $R = \mathbb{C}[x] \subseteq R' = \mathbb{C}[x,y]/(f)$ where $f(x,y) = xy - 1$. Consider ideals (x) and (x,y) of $\mathbb{C}[x,y]$. Since

$$\frac{\mathbb{C}[x,y]}{(x)} \cong \mathbb{C}[y] \qquad \frac{\mathbb{C}[x,y]}{(x,y)} \cong \mathbb{C}$$

are integral domains, the ideals (x) and (x,y) are prime ideals. Moreover, since $(xy) \subseteq (x) \subset (x,y)$, $\overline{(x)} \subset \overline{(x,y)}$ are prime ideals of $R' = \mathbb{C}[x,y]/(f)$. Moreover, note that:

$$\overline{(x)} \cap \mathbb{C}[x] = (x) = \overline{(x,y)} \cap \mathbb{C}[x]$$

Hence, the extension $R = \mathbb{C}[x] \subseteq R' = \mathbb{C}[x,y]/(f)$ is not an integral extension. Otherwise, this will contradict Proposition 29.18.

Corollary 29.20. ***(Going Up Theorem)*** *If $\varphi : R \to R'$ is an integral homomorphism and $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ are prime ideals in R, and $\mathfrak{q}_1$ is a prime ideal in R' such that $\varphi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$, then there is a prime ideal $\mathfrak{q}_2$ in R' with $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\varphi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$.*

Proof. Consider the induced homomorphism $g : R/\mathfrak{p}_1 \to R'/\mathfrak{q}_1$, which is injective and integral. Applying Lemma 29.14 to the prime ideal $\mathfrak{p}_2/\mathfrak{p}_1$ in $R/\mathfrak{p}_1$, we conclude that there is a prime ideal $\mathfrak{q}_2/\mathfrak{q}_1$ in $R'/\mathfrak{q}_1$ such that $g^{-1}(\mathfrak{q}_2/\mathfrak{q}_1) = \mathfrak{p}_2/\mathfrak{p}_1$. It is then clear that $\mathfrak{q}_2$ satisfies the conclusion going up theorem. □

30. Noether's Normalization & Hilbert's Nullstellensatz

Let us start by giving the geometric idea behind this so-called Noether Normalization theorem. Consider the coordinare ring

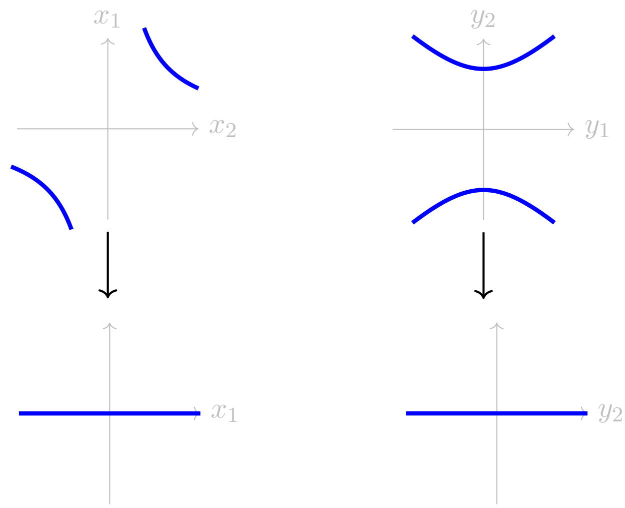
$$R = \mathbb{C}[x_1, x_2]/(x_1x_2 - 1)$$

of the affine algebraic set $X = \mathbb{V}(x_1x_2 - 1) \subset \mathbb{A}^2$. By Example 29.17, we know already that R is not integral (and hence not finite) over $\mathbb{C}[x_1]$. It is easy to change this, however, by a linear coordinate transformation: if we set, for example,

$$x_1 = y_2 + y_1, \quad x_2 = y_2 - y_1$$

then we can write R also as $R = \mathbb{C}[y_1, y_2]/(y_2^2 - y_1^2 - 1)$, and this is now finite over $\mathbb{C}[y_1]$ by Lemma 29.4 since the polynomial $y_2^2 - y_1^2 - 1$ is monic in y_2. Geometrically, the coordinate transformation has rotated $\mathbb{A}^2$ so that the Lying Over property now obviously holds. In terms of geometry, we are therefore looking for a change of coordinates so that a suitable coordinate projection to some affine space $\mathbb{A}^r$ corresponds to a finite ring extension of a polynomial ring over k in r variables.

Proposition 30.1. ***(Noether's Normalization Lemma)*** *Let R be a finitely generated as a $\mathbb{K}$-algebra with generators $x_1, \cdots, x_n$. For some positive integer $0 \leq r \leq n$, we can find $z_1, \cdots, z_r \in R$ such that there is an injective $\mathbb{K}$-algebra morphism $\mathbb{K}[z_1, \cdots, z_r] \to R$ such that R is a finite extension of $\mathbb{K}[z_1, \ldots, z_r]$.*



$f = x_1x_2 - 1$ $\qquad$ $f = y_2^2 - x_2^2 - 1$

The strategy is to find a suitable change of coordinates so that the given relations among the variables become monic. It turns out that a linear change of coordinates works in general only for infinite fields, whereas for arbitrary fields one has to allow more general coordinate transformations.

Lemma 30.2. *Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-zero polynomial over an infinite field $\mathbb{K}$. Assume that f is homogeneous, i.e., every monomial of f has the same degree. Then there are $a_1, \ldots, a_{n-1} \in k$ such that $f(a_1, \ldots, a_{n-1}, 1) \neq 0$.*

PROOF. The case $n = 1$ is trivial, since a homogeneous polynomial in one variable is just a constant multiple of a monomial. Assume $n > 1$. Write f as

$$f = \sum_{i=0}^{d} f_i x_1^i$$

where the $f_i \in \mathbb{K}[x_2, \ldots, x_n]$ are homogeneous of degree $d - i$. As f is non-zero, at least one f_i has to be non-zero. By induction, we can therefore choose $a_2, \ldots, a_{n-1}$ such that $f_i(a_2, \ldots, a_{n-1}, 1) \neq 0$ for this i. But then $f(\cdot, a_2, \ldots, a_{n-1}, 1) \in \mathbb{K}[x_1]$ is a non-zero polynomial, so it has only finitely many zeroes. As $\mathbb{K}$ is infinite, we can therefore find $a_1 \in \mathbb{K}$ such that $f(a_1, \ldots, a_{n-1}, 1) \neq 0$. □

Lemma 30.3. *Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-zero polynomial over an infinite field $\mathbb{K}$. Then there exist $\lambda \in \mathbb{K}$ and $a_1, \ldots, a_{n-1} \in k$ such that*

$$\lambda f(y_1 + a_1 y_n, y_2 + a_2 y_n, \ldots, y_{n-1} + a_{n-1} y_n, y_n) \in \mathbb{K}[y_1, \ldots, y_n]$$

is monic in y_n.

PROOF. Let d be the degree of f, and write

$$f = \sum_{k_1, \ldots, k_n} c_{k_1, \ldots, k_n} x_1^{k_1} \cdots x_n^{k_n} \qquad c_{k_1, \ldots, k_n} \in k$$

Then the leading term of

$$\lambda f(y_1 + a_1 y_n, y_2 + a_2 y_n, \ldots, y_{n-1} + a_{n-1} y_n, y_n)$$

$$= \lambda \sum_{k_1,\ldots,k_n} c_{k_1,\ldots,k_n}(y_1 + a_1 y_n)^{k_1} \cdots (y_{n-1} + a_{n-1}y_n)^{k_{n-1}} y_n^{k_n}$$

in y_n is obtained by always taking the second summand in the brackets and only keeping the degree-d terms, i.e., it is equal to

$$\lambda \sum_{k_1+\ldots+k_n=d} c_{k_1,\ldots,k_n} a_1^{k_1} \cdots a_{n-1}^{k_{n-1}} y_n^{k_1+\ldots+k_n} = \lambda f_d(a_1, \ldots, a_{n-1}, 1) y_n^d,$$

where f_d is the (homogeneous) degree-d part of f. Now pick $a_1, \ldots, a_{n-1}$ by Lemma 30.2 such that $f_d(a_1, \ldots, a_{n-1}, 1) \neq 0$, and set $\lambda = [f_d(a_1, \ldots, a_{n-1}, 1)]^{-1}$.

□

Lemma 30.4. *Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-zero polynomial over an arbitrary field $\mathbb{K}$. Then there exists $\lambda \in k$ and $a_1, \ldots, a_{n-1} \in \mathbb{N}$ such that*

$$\lambda f(y_1 + y_n^{a_1}, y_2 + y_n^{a_2}, \ldots, y_{n-1} + y_n^{a_{n-1}}, y_n) \in k[y_1, \ldots, y_n]$$

is monic in y_n.

Proof. (Proposition 30.1) We will prove the statement by induction on the number n of generators of R. The case $n = 0$ is trivial, as we can then choose $r = 0$ as well. Moreover, if there is no algebraic relation among the $x_1, \ldots, x_n \in R$, then

$$\mathbb{K}[x_1, \cdots, x_n] \cong R,$$

and the claim is clear in this case. Assume there is a non-zero polynomial f over k such that $f(x_1, \ldots, x_n) = 0$ in R. Choose λ and $a_1, \ldots, a_{n-1}$ as in Lemma 30.3 (if k is infinite) or Lemma 30.4 (for any k) and set

$$y_1 := x_1 - a_1 x_n, \cdots, y_{n-1} := x_{n-1} - a_{n-1}x_n, y_n := x_n \qquad \text{or}$$
$$y_1 := x_1 - x_n^{a_1}, \ldots, y_{n-1} := x_{n-1} - x_n^{a_{n-1}}, y_n := x_n$$

respectively, In both cases, these relations show that the $\mathbb{K}$-subalgebra $\mathbb{K}[y_1, \ldots, y_n]$ of R generated by $y_1, \ldots, y_n \in R$ is the same as that generated by $x_1, \ldots, x_n$, i.e., all of R. Moreover, y_n is integral over the $\mathbb{K}$-subalgebra $k[y_1, \ldots, y_{n-1}]$ of R, since

$$\lambda f(y_1 + a_1 y_n, \ldots, y_{n-1} + a_{n-1}y_n, y_n) \qquad \text{or}$$
$$\lambda f(y_1 + y_n^{a_1}, \ldots, y_{n-1} + y_n^{a_{n-1}}, y_n)$$

respectively, is monic in y_n and equal to $\lambda f(x_1, \ldots, x_n) = 0$. Hence $R = k[y_1, \ldots, y_n]$ is finite over $\mathbb{K}[y_1, \ldots, y_{n-1}]$ by Lemma 29.4. In addition, the subalgebra $k[y_1, \ldots, y_{n-1}]$ of R is finite over a polynomial ring $\mathbb{K}[z_1, \ldots, z_r]$ by the induction hypothesis, and thus R is finite over $\mathbb{K}[z_1, \ldots, z_r]$ by transitivity. □

Example 30.5. Let $R = \mathbb{C}[x_1, x_2]/(x_1x_2 - 1)$. Consider $f(x_1, x_2) = x_1x_2 - 1 = 0$. Then:

$$f(\overline{x_1}, \overline{x_2}) = \overline{x_1 x_2} - 1 = 0$$

The homogeneous degree 2 term is $g(x_1, x_2) = x_1x_2$. Then for any $a_1 \in \mathbb{C}^*$, we have that $g(a_1, 1) = a_1 \neq 0$. Set $\lambda = [g(a_1, 1)]^{-1} = 1/a_1$. Consider

$$y_1 = x_1 - a_1 x_2 \qquad y_2 = x_2$$

Then:

$$0 = f(\overline{x_1}, \overline{x_2}) = \lambda f(\overline{y_1} + a_1\overline{y_1}, \overline{y_2}) = \frac{1}{a_1}\left(\overline{y_2}(\overline{y_1} + a_1\overline{y_2}) - 1\right) = \overline{y_2}^2 + \frac{1}{a_1}\overline{y_1 y_2} - \frac{1}{a_1}$$

Therefore $\overline{y_1}$ is a monic polynomial in $\mathbb{C}[\overline{y_1}]$. Choosing $a_1 = 1$, we have that $R = \mathbb{C}[x_1, x_2]/(x_1x_2 - 1)$ is finite over $\mathbb{C}[\overline{x_1 - x_2}]$

Using these results, we would like to state and prove Hilbert's Nullstellensatz. First, we prove Zariski's lemma.

Lemma 30.6. ***(Zariski's Lemma)*** *Let $\mathbb{K}$ be a field, and let R be a finitely generated $\mathbb{K}$-algebra which is also a field. Then $\mathbb{K} \subseteq R$ is a finite field extension. In particular, if in addition $\mathbb{K}$ is algebraically closed, then $R = \mathbb{K}$.*

Proof. By Proposition 30.1, we know that R is finite extension over a polynomial ring $\mathbb{K}[z_1, \ldots, z_r]$, and thus also integral over $\mathbb{K}[z_1, \ldots, z_r]$ since finite ring extensions are integral. But R is a field, hence $\mathbb{K}[z_1, \ldots, z_r]$ must be a field as well by Lemma 29.14(1). This is only the case for $r = 0$, and so R is finite over k. □

Proposition 30.7. ***(Hilbert's Nullstellensatz)*** *Let $\mathbb{K}$ be an algebraically closed field. The maximal ideals of $\mathbb{K}[x_1, \cdots, x_n]$ are of the form $(x_1 - a_1, \cdots, x_n - a_n)$.*

Proof. Clearly, ideals of the form $(x_1 - a_1, \cdots, x_n - a_n)$ are maximal. This is because:

$$\frac{\mathbb{K}[x_1, \cdots, x_n]}{(x_1 - a_1, \cdots, x_n - a_n)} \cong \mathbb{K}.$$

Conversely, let $\mathfrak{m}$ be any maximal ideal in $\mathbb{K}[x_1, \cdots, x_n]$. Then the quotient ring,

$$R = \frac{\mathbb{K}[x_1, \cdots, x_n]}{\mathfrak{m}},$$

is a field, which is also a finitely generated $\mathbb{K}$-algebra. By Lemma 30.6, R is a field, and $\mathbb{K} \subseteq R$ is a finite field extension. Since $\mathbb{K}$ is algebraically closed, $R = \mathbb{K}$. □

Part 6. References

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