CATEGORY THEORY

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Abstract. These notes cover various topics in category theory. I wrote these notes during graduate school. References used include [Rie17; Lei14; Mac13; Alu21]. There may be typos and errors. Please send corrections to junaid.aftab1994@gmail.com.

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Part 1. General Theory

We begin by introducing categories and functors, which provide a unifying framework for many mathematical structures. We then study representable functors and the Yoneda Lemma, which offers deep insight into how objects are determined by their relationships to others.

1. Categories

In mathematics, we consider various types of structures - groups, rings, topological spaces, smooth manifolds, Hilbert spaces etc. There are two basic ingredients: objects having the desired structure and maps between objects which preserve the structure. The concept of a category axiomatizes this idea. The goal of this section is to introduce some basic notions in category theory.

Definition 1.1. A **category** $\mathscr{C}$ consists of a collection of **objects**, $\mathrm{Obj}(\mathscr{C})$. For every $X, Y \in \mathrm{Obj}(\mathscr{C})$, there is a collection $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of **morphisms**. For every $X, Y, Z \in \mathscr{C}$, there exists a composition map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z) \qquad (f, g) \mapsto g \circ f$$

and a unit $\mathrm{Id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ satisfying the following axioms:

(1) The Hom sets are pairwise disjoint; that is, each $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ has a unique **domain** X and a unique **target** Y.
(2) **(Identity Axiom)** For each $X \in \mathrm{Obj}(\mathscr{C})$, there is an **identity morphism** $\mathrm{Id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ called the **identity morphism**, which has the property that

$$f \circ \mathrm{Id}_X = f \qquad \mathrm{Id}_Y \circ f = f$$

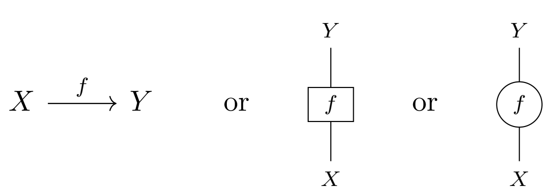
for every $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.
(3) **(Associativity Axiom)** Composition is **associative** whenever defined. That is, given $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y), g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$ and $h \in \mathrm{Hom}(Z, Z')$, one has

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Remark 1.2. *If $X \in \mathrm{Obj}\,(\mathscr{C})$, we simply say $X \in \mathscr{C}$. If the categeory, $\mathscr{C}$, is clear from context, we write $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ as simply $\mathrm{Hom}(X, Y)$.*

Remark 1.3. *Objects of a category are in bijective correspondence with the identity morphisms, which are uniquely determined by the property that they serve as two-sided identities for composition.*

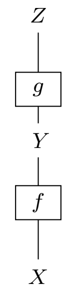
A common practice is to diagrammatically represent objects and morphisms in a category. For instance, objects in a category are represented as labels such as X, Y, Z etc. and morphisms between objects are drawn as arrows between labels. For instance, a $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is drawn as:



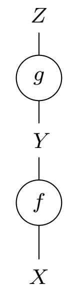
The first diagram should be read from left to right, while the second and third diagrams should be read from bottom to top. We will utilize all diagrammatic notations as needed. Similarly, composition between $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$ is drawn as

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

or



or



Example 1.4. (**Familiar Examples**) A category is **concrete** if the objects have underlying sets and whose morphisms are functions between these underlying sets. The following are examples of some familiar concrete categories:

(1) Sets: The prototypical example is the category of sets, denoted Sets. In this category, the objects are sets, and $\mathrm{Hom}(X, Y)$ is the set of functions from X to Y.
(2) Sets_*: The objects of Sets_* are pointed sets, (X, x_0), which are sets with a distinguished point as objects and morphisms are functions that preserve the distinguished point.
(3) Grp: The objects of Grp are groups, and $\mathrm{Hom}(X, Y)$ is the set of all group homomorphisms from X to Y.
(4) Rings: The objects of Rings are rings, and $\mathrm{Hom}(X, Y)$ is the set of all rings homomorphisms from X to Y.
(5) Vec_k: The objects of Vec_k are vector spaces over a given field, k, and $\mathrm{Hom}(X, Y)$ is the set of all linear transformations from X to Y.
(6) ${}_R\mathsf{Mod}$: For a ring, R, the objects of ${}_R\mathsf{Mod}$ are left R-modules over a given ring, R, and $\mathrm{Hom}(X, Y)$ is the set of all R-module homomorphisms from X to Y. When $R = k$, a field, it corresponds to Vec_k. When $R = \mathbb{Z}$, it corresponds to Ab, the category of abelian groups.
(7) Mod_R: The objects of Mod_R are right R-modules over a given ring, R, and $\mathrm{Hom}(X, Y)$ is the set of all R-module homomorphisms from X to Y.
(8) Top: The objects of Top are topological spaces, and $\mathrm{Hom}(X, Y)$ is the set of continuous functions from X to Y.
(9) Top_*: The objects of Top_* are topological spaces with a distinguished point, and the morphisms are continuous functions that preserve the distinguished point.

Remark 1.5. *Russell's paradox demonstrates that there cannot be a set containing 'all sets.' To address this, we have adopted the broader term 'collection' in defining a category. As category theory develops, the foundational set-theoretical challenges become increasingly pronounced. To navigate these issues, category theorists often extend the standard Zermelo–Fraenkel axioms of set theory by introducing axioms that distinguish between 'small' and 'large' sets, or between sets and classes. However, for simplicity, we disregard such foundational concerns in these notes.*

Definition 1.6. We list a number of relevant definitions below:

(1) $\mathscr{C}'$ is a **subcategory** of $\mathscr{C}$ and we write $\mathscr{C}' \subseteq \mathscr{C}$ if:
  - $\mathrm{Ob}(\mathscr{C}')$ is a sub-collection of $\mathrm{Ob}(\mathscr{C})$;

- For all $X, Y \in \mathrm{Ob}(\mathscr{C}')$, we have $\mathrm{Hom}_{\mathscr{C}'}(X,Y)$ is a sub-collection of $\mathrm{Hom}_{\mathscr{C}}(X,Y)$;
- The composition of $\mathscr{C}'$ is the restriction of the composition of $\mathscr{C}$.

$\mathscr{C}'$ is additionally a **full subcategory** if, for all $X, Y \in \mathrm{Ob}(\mathscr{C}')$, we have $\mathrm{Hom}_{\mathscr{C}'}(X,Y)$ coincides with $\mathrm{Hom}_{\mathscr{C}}(X,Y)$.

(2) A category, $\mathscr{C}$, is called **discrete** if it contains no morphisms apart from identity morphisms.

(3) A category, $\mathscr{C}$, is called **small** if both $\mathrm{obj}(\mathscr{C})$ and $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ are sets for each $X, Y \in \mathscr{C}$.

(4) A category, $\mathscr{C}$, is **locally small** if $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is a set for each $X, Y \in \mathscr{C}$.

(5) Let $\mathscr{C}$ be a category. An **isomorphism** in a category is a morphism $f \in \mathrm{Hom}(X,Y)$ for which there exists a morphism $g \in \mathrm{Hom}(Y,X)$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. The objects X and Y are isomorphic whenever there exists an isomorphism between X and Y, in which case one writes $X \simeq Y$.

(6) A category, $\mathscr{C}$, is a **groupoid** if every morphism is an isomorphism. Note that every category, $\mathscr{C}$, contains a **maximal groupoid**, the subcategory containing all of the objects and only those morphisms that are isomorphisms.

Example 1.7. Here are some instances of full subcategories:

(1) The full subcategory of Sets whose objects are finite sets is called the category of finite sets and is denoted by $\mathsf{Sets}_{\mathsf{Fin}}$.

(2) The category of finite-dimensional k-vector space, $\mathsf{Vec}_k^{\mathsf{Fd}}$, is a full subcategory of the category of k-vector spaces , Vec_k.

(3) The category of abelian groups, Ab, is a full subcategory of the category of groups, Grp.

(4) The category of commutative rings, CRings, is a full subcategory of the category of rings, Rings.

Example 1.8. The following is a characterization of isomorphisms in various categories:

(1) The isomorphisms in Sets are precisely the bijections.

(2) The isomorphisms in $\mathsf{Group}/\mathsf{Ring}$, are bijective group/ring isomomorphisms.

(3) The isomorphisms in the category Top are homeomorphisms, i.e., the continuous functions with continuous inverses. Note that, in contrast to the situation in Grp and Rings, a bijective continuous map in Top is not necessarily an isomorphism. A classic example is the map

$$\begin{aligned} [0,1) &\to \mathbb{S}^1, \\ t &\mapsto e^{2\pi i t}. \end{aligned}$$

which is a continuous bijection but not a homeomorphism.

Remark 1.9. *Let $\mathscr{C}$ be a category. Here are some basic properties of morphisms $\mathscr{C}$:*

(1) *Consider a morphism $f \in \mathrm{Hom}(X,Y)$. Assume there exists $g, h \in \mathrm{Hom}(Y,X)$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ h = \mathrm{Id}_Y$. Note that we have,*

$$g = g \circ \mathrm{Id}_Y = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{Id}_X \circ h = h,$$

Hence, $g = h$. Note that this readily implies that f is an isomorphism.

(2) *Let $f \in \mathrm{Hom}(X,Y)$ be an isomorphism such that there exist $g, h \in \mathrm{Hom}(Y,X)$ such that both g and h are inverse isomorphism for f. In particular, we have $g \circ f = \mathrm{Id}_X$ and $f \circ h = \mathrm{Id}_Y$. By (1) $g = h$. Hence, f can have at most one inverse morphism.*

(3) *Consider three morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

Assume $g \circ f$ *and* $h \circ g$ *are isomorphisms. Since* $g \circ f$ *is an isomorphism, there exists a morphism* $(g \circ f)^{-1} : Z \to X$ *such that*

$$(g \circ f) \circ (g \circ f)^{-1} = \mathrm{Id}_Z \quad \textit{and} \quad (g \circ f)^{-1} \circ (g \circ f) = \mathrm{Id}_X .$$

Similarly, since $h \circ g$ *is an isomorphism, there exists a morphism* $(h \circ g)^{-1} : W \to Y$ *such that*

$$(h \circ g) \circ (h \circ g)^{-1} = \mathrm{Id}_W \quad \textit{and} \quad (h \circ g)^{-1} \circ (h \circ g) = \mathrm{Id}_Y .$$

Note that we have

$$(h \circ g \circ f) \circ ((g \circ f)^{-1} \circ g \circ (h \circ g)^{-1}) = \mathrm{Id}_W,$$
$$((g \circ f)^{-1} \circ g \circ (h \circ g)^{-1}) \circ (h \circ g \circ f) = \mathrm{Id}_X .$$

This shows that $h \circ g \circ f$ *is an isomorphism. Since* $h \circ g$ *is an isomorphism,* f *is an isomorphism as well.*

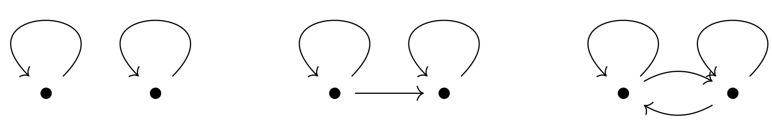
In all of the above examples, the objects of the categories were in obvious ways sets with additional structure. This need not be the case, as the next example shows.

Example 1.10. A pre-ordered set $(S, \leq)$ consits of a set, S, equipped with a binary relation $\leq$ on S that satisfies the following properties:

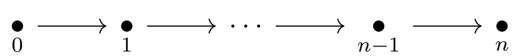
(1) (**Reflexivity**) $x \leq x$
(2) (**Transitivity**) $x \leq y$ and $y \leq z$ imply $x \leq z$

A pre-ordered $(S, \leq)$ can be interpreted as a category whose objects are the elements of S, and there is a single morphism from x to y if and only if $x \leq y$ (and no morphism otherwise). he following are examples of such categories:

(1) Let $S = \{\bullet\}$. This is a pre-ordered set and the the corresponding category is with one object and one identity arrow.
(2) Let $S = \{*_1, *_2\}$ be a two-element set. There are three pre-orderes on S corresponding to the categories shown below:

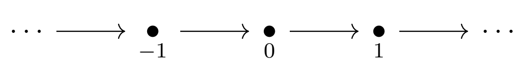


(3) For some $n \in \mathbb{N}$, consider the partially ordered set whose elements are $S = \{0, 1, \cdots, n-1\}$ with the usual partial order that $0 \leq 1 \leq \cdots \leq n-1$. By the example above, this is a category which can be visualized as follows:



Identity morphisms, which can be depicted by loops, are not drawn in the diagram above.

(4) Consider $S = \mathbb{Z}$, the set of integers, to be the partially ordered set with the usual partial order. This also forms a category which can be visualized as follows:



Identity morphisms, which can be depicted by loops, are not drawn in the diagram above.

(5) If X is a topological space, then the open subsets form a partially ordered set, where the order is given by inclusion. Informally, if $U \subseteq V$, then we have exactly one morphism $U \to V$ in the category (and otherwise none). We write the corresponding category as $\mathscr{O}(X)$.

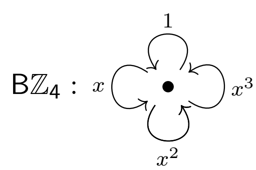
Remark 1.11. *An ordered set is a pre-ordered set S satisfying the anti-symmetry axiom:*

$$\text{If } x \leq y \text{ and } y \leq x, \text{ then } x = y \text{ (antisymmetry).}$$

Note that the only isomorphisms in an ordered set are the identity morphisms.

Example 1.12. (**Less familiar examples of a category**) The following are examples of some 'lesser known' categories. We will mention these categories from time to time for illustrative purposes.

(1) Let R be a ring. Then Mat_R is the category whose objects are positive integers and in which the set of morphisms from n to m is the set of $m \times n$ matrices with values in R. Composition is by matrix multiplication with identity matrices serving as the identity morphisms.
(2) Let G be a group. The category BG is the category whose set of objects is a singleton $\{*\}$ and such that $\mathrm{Hom}_{\mathsf{BG}}(*, *) = G$; the composition law is given by the multiplication of G. Note that BG is a groupoid.



(3) Ch_R has chain complexes of R-modules as objects and chain maps as morphisms.
(4) Htpy, like Top, has spaces as its objects but morphisms are homotopy classes of continuous maps.
(5) Htpy_* has based spaces as its objects and basepoint-preserving homotopy classes of based continuous maps as its morphisms.
(6) Let X be a topological space. The fundamental groupoid, $\Pi_1(X)$, of X is the category whose set of objects is X and such that, for $x, y \in X$, the set of morphisms from x to y is the set of continuous maps $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$ modulo homotopies fixing the endpoints (i.e., continuous paths from x to y modulo homotopy). The composition law of $\Pi_1(X)$ is given by concatenation of paths. Note that $\Pi_1(X)$ is a groupoid

We conclude this section with an important construction: the product of two categories:

Definition 1.13. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The **product category**, $\mathscr{C} \times \mathscr{D}$, is defined such that:

- $\mathrm{Ob}(\mathscr{C} \times \mathscr{D})$ are ordered pairs (X, Y), where $X \in \mathscr{C}$ and $Y \in \mathscr{D}$,
- For all $(X, Y), (X', Y') \in \mathrm{Ob}(\mathscr{C} \times \mathscr{D})$, we have $\mathrm{Hom}_{\mathscr{C} \times \mathscr{D}}((X, Y), (X', Y'))$ are ordered pairs (f, g) where $f \in \mathrm{Hom}_{\mathscr{C}}(X, X')$ and $g \in \mathrm{Hom}_{\mathscr{D}}(Y, Y')$.
- Composition and identities are defined componentwise.

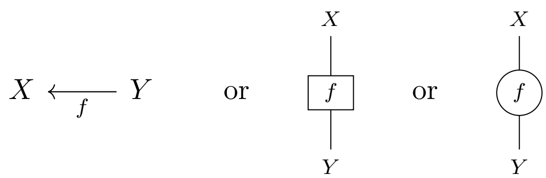
2. Principle of Duality

A category is a mathematical object in its own right. Therefore, one can ask the question: how can one construct new categories out of old categories? If we visualize the morphisms in a category as arrows pointing from their domain object to their codomain object, we might imagine simultaneously reversing the directions of every arrow.

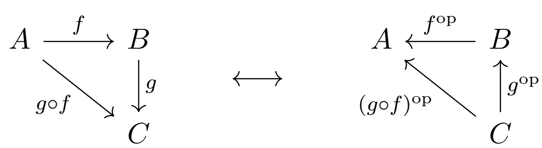
Definition 2.1. Let $\mathscr{C}$ be a category. The **opposite category**, $\mathscr{C}^{\mathrm{op}}$, is defined by

$$\mathrm{Obj}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathscr{C})$$
$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \{f^{\mathrm{op}} \mid f \in \mathrm{Hom}_{\mathscr{C}}(B, A)\}$$

If $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, then $f^{\mathrm{op}} \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y)$ is graphically denoted as[1]



Remark 2.2. *It is an easy matter to check that $\mathscr{C}^{\mathrm{op}}$ is indeed a category. For instance, for each $X \in \mathscr{C}$, the arrow $\mathrm{Id}_X^{\mathrm{op}}$ serves as its identity in $\mathscr{C}^{\mathrm{op}}$. Moreover, composition of morphisms is defined as follows. The morphism*



The associativity and identity axioms are clear because the analog of these axioms hold in $\mathscr{C}$.

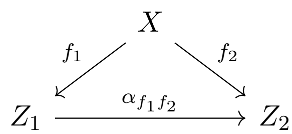
Example 2.3. The following are examples of opposite categories:

1. $\mathsf{Mat}_R^{\mathrm{op}}$ is the category whose objects are non-zero natural numbers and in which a morphism from m to n is an $m \times n$ matrix with values in R.
2. The opposite category of $(S, \leq)$ is $(S, \leq')$ where the binary relation $\leq'$ is such that $x \leq' y$ if and only if $y \leq x$.
3. If we identify a group, G, with the category $\mathsf{B}G$, then $(\mathsf{B}G)^{\mathrm{op}}$ is a category with one object such that each morphism is an isomorphism. Hence, $(\mathsf{B}G)^{\mathrm{op}}$ can be identified with a group. We write $(\mathsf{B}G)^{\mathrm{op}} \simeq \mathsf{B}G^{\mathrm{op}}$ The group G^{op} is called the opposite group. It can be identified as a group with the multiplication rule given as
$$g *_{G^{\mathrm{op}}} h = h *_G g, \qquad g, h \in G$$

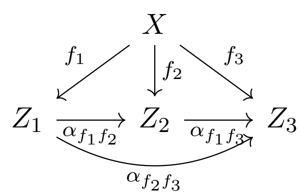
The definition of an opposite category may seem artificial. However, it underpins a key principle in modern mathematics: duality. If $\mathscr{C}$ is a category, then $\mathscr{C}^{\mathrm{op}}$ is a category and it is easy to see that $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$ is the category $\mathscr{C}$. This suggests that one move back and forth between a category and its opposite category. This duality has significant implications. For instance, any theorem that holds in a category $\mathscr{C}$ extends to a dual theorem in $\mathscr{C}^{\mathrm{op}}$. In order to put this idea to work, we discuss an example where we construct a new category and then apply the duality principle to construct another category.

[1]The second and third diagrams are to be read from top to bottom.

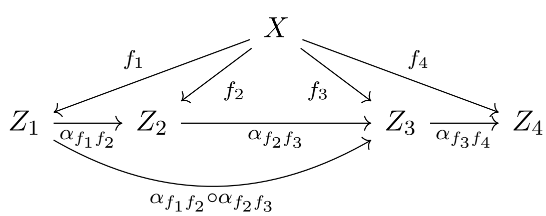
Example 2.4. (Slice Category) Let $\mathscr{C}$ be a category and let $X \in \mathscr{C}$. The X**-slice category**, denoted as $\mathscr{C}^X$, is a category whose objects are morphisms $f : X \to Z_1$ with target $Z_1 \in \mathscr{C}$, and in which a morphism from $f_1 : X \to Z_1$ to $f_1 : X \to Z_2$ is a map $\alpha_{f_1 f_2} : Z_1 \to Z_2$ such that the diagram

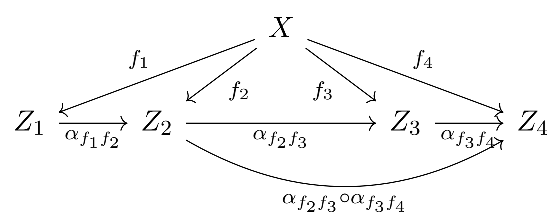


commutes. Let's verify that this is indeed a category. If $f_1 : X \to Z_1$, $f_2 : X \to Z_2$ and $f_3 : X \to Z_3$ are objects in $\mathscr{C}^X$, then morphsisms $\alpha_{f_1 f_2}$ and $\alpha_{f_2 f_3}$ can be composed to define a morphism $\alpha_{f_1 f_3} = \alpha_{f_1 f_2} \circ \alpha_{f_2 f_3}$ since these morphisms can be composed in $\mathscr{C}$. As a result, the following diagram

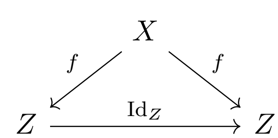


commutes in $\mathscr{C}$. The associativity axiom holds in $\mathscr{C}^X$ as well. This is best seen visually since the following two diagrams

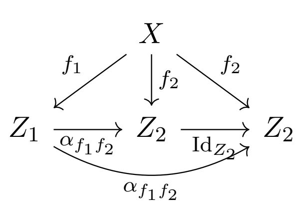


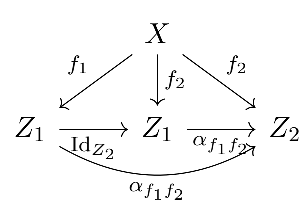


are the same via the associativity axiom in $\mathscr{C}$. For an object $f : X \to Z$, the identity morphism is simply represented by the morphism $\text{Id}_Y : Z \to Z$ such that the diagram



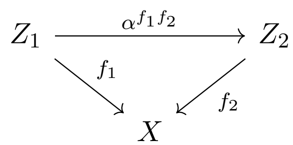
commutes and the identity axiom holds in $\mathscr{C}^X$ since the following two diagram





commutes in $\mathscr{C}$. An example of a slice category is Top_*, where $\{*\}$ is the one-point space.

Example 2.5. (Coslice Category) Let $\mathscr{C}$ be a category and let $X \in \mathscr{C}$. The X**-coslice category**, denoted as $\mathscr{C}_X$, is a category whose objects are morphisms $f : Z_1 \to X$ with domain $Z_1 \in \mathscr{C}$, and in which a morphism from $f_1 : Z_1 \to X$ to $f_2 : Z_2 \to X$ is a map $\alpha_{fg} : Z_1 \to Z_2$ such that the diagram



commutes. It turns out that we don't need to go through the pain of verifying that this is indeed a well-defined category since the category $\mathscr{C}_X$ is simply $((\mathscr{C}^{\mathrm{op}})^X)^{\mathrm{op}}$, which we know to be a category.

Let's now invoke the principle of duality to study properties of morphisms in a category.

Proposition 2.6. *Let $\mathscr{C}$ be a category. The following are equivalent:*

(1) *$f : X \to Y$ is an isomorphism in $\mathscr{C}$.*
(2) *For all objects $Z \in \mathscr{C}$, post-composition with f defines a bijection $f_* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$.*
(3) *For all objects $Z \in \mathscr{C}$, pre-composition with f defines a bijection $f^* : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$.*

Proof. We prove the equivalence of (1) and (2). Assume $f : X \to Y$ is an isomorphism with inverse $g : Y \to X$, then, post-composition with g defines an inverse function

$$g_* : \mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X)$$

for f_*. Indeed, for any $h : Z \to X$ and $k : Z \to Y$,

$$g_* \circ f_*(h) = g \circ f \circ h = h,$$

and

$$f_* \circ g_*(k) = f \circ g \circ k = k.$$

Conversely, assuming (2), there must be an element $g \in \mathrm{Hom}(Y, X)$ whose image under f_* is Id_Y. By construction, $\mathrm{Id}_Y = f \circ g$. By associativity of composition, the elements $g \circ f, \mathrm{Id}_X \in \mathrm{Hom}(X, X)$ have the common image f under the function $f_* : \mathrm{Hom}(X, X) \to \mathrm{Hom}(X, Y)$, whence $g \circ f = \mathrm{Id}_X$. Thus, f and g are inverse isomorphisms. The equivalence of (1) and (3) follows by a duality argument. □

Remark 2.7. *In the language of representable functors introduced later, Proposition 2.6 shows that isomorphisms in a locally small category can be described representably through isomorphisms in* Sets*. Specifically, a morphism $f : X \to Y$ in an arbitrary locally small category $\mathscr{C}$ is an isomorphism if and only if the post-composition function $f_* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ between hom-sets defines an isomorphism in* Sets *for each object $Z \in \mathscr{C}$. Similar remarks as above apply to the that the pre-composition function.*

Example 2.8. Here is another example of the principle of duality. Let $(S, \leq)$ be a poset category. We can define the notion of an upper bound and a lower bound of a sub-collection of objects, A, in $(S, \leq)$ as follows. First, we say that $U \in S$ is an upper bound of A if

$$X \leq U \quad \Longleftrightarrow \quad \text{There exists a morphism in}\, \mathrm{Hom}(X, U),$$

for all objects X in A. A lower bound of A is defined to be an upper bound of A in the opposite category, $(S, \leq')$. In other words, we say that $L \in S$ is a lower bound of A if

$$L \leq X \quad \Longleftrightarrow \quad \text{There exists a morphism in}\, \mathrm{Hom}(L, X),$$

for all objects X in A. Let F denote be the collection of all upper bounds of A. The supremum of A, if it exists, is defined to be a lower bound of F contained in F. The infimum of A, if it exists, is defined to be the dual of the supremum.

Remark 2.9. *If $(S, \leq)$ is a partially ordered set, the condition of containment implies that the supremum, if it exists, is unique. Suppose we have two lower bounds L and L' of F. Then $L \leq L'$ and $L' \leq L$ since both are contained in L. Since $(S, \leq)$ is partially ordered, we have that $L = L'$.*

2.1. **Monomorphisms and Epimorphisms.** Before concluding this section, we discuss some important examples of morphisms in a category and once again invoke the principle of duality to prove properties about these special morphisms. In an arbitrary category, the notion of an injective and surjective morphisms may not make sense; after all, morphisms in a category need not be functions. On the other hand, it is often convenient to have purely categorical understanding of injective and surjective maps, which allows us to make a categorical definition of such notions in an arbitrary category. This leads to the definition of monomorphism and epimorphisms.

Definition 2.10. Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ is a **monomorphism** (or is **monic**) if f can be canceled from the left; that is, for all objects Z and all morphisms $g, h : Z \to X$, we have that $fg = fh$ implies $g = h$.

$$Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

Example 2.11. Consider Fld, the category of fields. Let $f \in \mathrm{Hom}_{\mathsf{Fld}}(X, Y)$. Its kernel is an ideal in X. Since X is a field, there are only two ideals: $\{0\}$ and X itself. The kernel of cannot be X since this would be the zero morphism which is not a field morphism. Thus, $\ker f = \{0\}$, and from this, f is injective, and in particular, it is left cancellable. If $g, h \in \mathrm{Hom}_{\mathsf{Fld}}(Z, X)$ such that $g \circ f = h \circ f$, then $h = k$ by the above argument f being left cancellable. This shows that every morphism in Fld is a monomorphism!

Proposition 2.12. *In a concrete category, an injective morphism is a monomorphism. In* Sets*, a monomorphism is an injective morphism.*

Proof. Suppose that $f : X \to Y$ is injective. Suppose further that we have mappings $g, h : Z \to X$ such that $g \neq h$. Then necessarily there exists some $z \in Z$ such that $g(z) \neq h(z)$. As f is injective, it follows that:

$$f(g(z)) \neq f(h(z))$$

Hence, $f \circ g \neq f \circ h$, implying that f is a monomorphism.

In Sets, assume f is a monomorphism. Let $x, x' \in X$ such that $x \neq x'$ and $Z = \{*\}$. Define:

$$g : \{*\} \to X, \ g(*) := x$$
$$h : \{*\} \to X, \ g(*) := x'$$

In particular, $g \neq h$. Since f is a monomorphism, $f \circ g \neq f \circ h$. It follows that it must be that

$$f(x) \neq f(x')$$

Hence f is injective. $\square$

Remark 2.13. *The converse of Proposition 2.12 is true in any category in which a singleton object exists such that an arbitrary map $* \to x$ is a morphism in the category. In particular, the converse of Proposition 2.12 is true in* Top[2]$, \mathsf{Grp}, \mathsf{Ring}$ [3] [4]. *However, the converse of Proposition 2.12 need not be true in an arbitrary concrete category. Let $\mathscr{C} = \mathsf{Div}$ be the full subcategory of Ab of divisible groups. The groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisble, and the canonical projection map $f : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is not injective, but we show that α is a monomorphism. Let $g, h : A \to \mathbb{Q}$ be morphisms and assume that $g \neq h$. Then $g(a) - h(a) = \frac{r}{s}$ for some $a \in A$ and $r, s \in \mathbb{Z}$ with $r \neq 0$ and $s \neq 0$. Since A is divisible, there exists $b \in A$ such that $a = nb$, where $n = 2r$. Then*

$$n[g(b) - h(b)] = g(nb) - h(nb) = \frac{r}{s},$$

so $g(b) - h(b) = \frac{1}{2s} \notin \mathbb{Z}$. Therefore, $f \circ g \neq f \circ h$, and the claim follows. Note that the proof given in Proposition 2.12 breaks down since a generic map $* \to x$ or $* \to x'$ need to be a group homomorphism.

Definition 2.14. Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ is an **epimorphism** (or is **epic**) if f can be canceled from the right; that is, for all objects Z and all morphisms $g, h : Y \to Z$, we have that $gf = hf$ implies $g = h$.

$$X \xrightarrow{\ f\ } Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

Proposition 2.15. *In a concrete category, a surjective morphism is an epimorphism. In* Sets*, an epimorphism is a surjective morphism.*

Proof. Suppose that $f : X \to Y$ is surjective. Suppose further that we have mappings $g, h : Y \to Z$ such that $g \neq h$. Then necessarily there exists some $y \in Y$ such that $g(y) \neq h(y)$. As f is surjective, there is a $x \in X$ such that $f(x) = y$. Hence,

$$g(f(x)) \neq g(f(x))$$

Hence, $g \circ f \neq g \circ f$, implying that f is an epimormphism.

In Sets, assume f is a epimormphism. Let $g : Y \to \{0, 1\}$ be the characteristic function of $\operatorname{Im} \alpha$, and let $h : Y \to \{0, 1\}$ be constantly 1. Then $g \circ \alpha = h \circ \alpha$ (both sides are constantly 1), so $g = h$ since f is an epimormphism. It follows that $\operatorname{Im} \alpha = Y$, that is, f is surjective. □

Remark 2.16. *The converse of Proposition 2.15 is true in some other concrete categories as well. For example, the converse of Proposition 2.15 is true in* $\mathsf{Top}, \mathsf{Grp}$[5]. *The converse of Proposition 2.15 need not be true in an arbitrary concrete category. For example, consider*

---

[2]

[3] The same proof as in Sets works for Top by noting that we can down $\{*\}$ with the discrete topology to ensure that an arbitrary map $* \to x$ is continuous.

[4] A slightly different proof is required in Grp and Ring. Let $\alpha : G \to H$ be a monomorphism in Grp. Let K be the kernel of α, and let $\beta_1 : K \to G$ and $\beta_2 : K \to G$ be the inclusion map and the trivial map, respectively. Then $\alpha \circ \beta_1 = \alpha \circ \beta_2$ since each side equals the trivial map $K \to H$. Since α is monomorphism, it follows that $\beta_1 = \beta_2$, which is to say that the inclusion map $K \to G$ is trivial. This forces K to be trivial, which says that α is injective. The proof for Ring is similar.

[5] The same proof as in Sets works for Top by noting that we can down $\{0, 1\}$ with the indiscrete topology to ensure that the indicator function is continuous. A slightly different proof is required for Grp. Details are skipped.

Ring. *The inclusion* $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ *is an epimorphism*[6] *in* Ring. *If* R *is a ring and* $f_1, f_2 : \mathbb{Q} \to R$ *are ring morphisms such that* $f_1 \circ \iota = f_2 \circ \iota$, *then* $f_1 = f_2$ *when restricted to* $\mathbb{Z} \subseteq \mathbb{Q}$. *But then:*

$$\begin{aligned} f_1(a/b) &= f_1(a)f_1(b^{-1}) \\ &= f_1(a)f_1(b)^{-1} \\ &= f_2(a)f_2(b)^{-1} \\ &= f_2(a)f_2(b^{-1}) = f_2(a/b) \end{aligned}$$

implies that $f_1 = f_2$. *However,* ι *is not surjective.*

Since the notions of monomorphism and epimorphism are dual, their abstract categorical properties are also dual, such as exhibited by the following proposition.

Proposition 2.17. *Let* $\mathscr{C}$ *be a category. The following are properties of monomorphisms in* $\mathscr{C}$*:*

(1) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are monomorphisms, then so is* $g \circ f : X \to Z$.
(2) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are morphisms so that* $g \circ f$ *is a monomorphism, then* f *is a monomorphism.*

Dually, the following are properties of epimorphisms $\mathscr{C}$*:*

(3) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are epimorphisms, then so is* $g \circ f : X \to Z$.
(4) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are morphisms so that* $g \circ f$ *is an epimorphism, then* g *is is an epimorphism.*

Proof. The proof proceeds as follows:

(1) Assume f, g are monomorphisms. Consider $h_1, h_2 : A \to X$ such that

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2$$

Note that

$$g \circ (f \circ h_1) = g \circ (f \circ h_2)$$

Since g is a monomorphism, $f \circ h_1 = f \circ h_2$. Since f is a monomorphism, $h_1 = h_2$. Hence, $g \circ f$ is a monomorphism.

(2) Assume $g \circ f$ is a monomorphism. Consider $h_1, h_2 : A \to X$ such that $h_1 \circ f = h_2 \circ f$. Composing by g on the right, we have that

$$h_1 \circ (f \circ g) = h_2 \circ (f \circ g)$$

Since $f \circ g$ is a monomorphism, we must have that $h_1 = h_2$. Hence, f is a monomorphism.

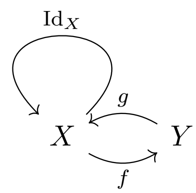
Since a monomorphism in $\mathscr{C}$ is an epimorphism in $\mathscr{C}^{\text{op}}$, the statements about epimorphisms hold by invoking the principle of duality. □

[6]It is also an monomorphism. Indeed, if R is a ring and $f_1, f_2 : R \to \mathbb{Z}$ are ring morphisms such that $\iota \circ f_1 = \iota \circ f_2$ then $f_1 = f_2$ necessarily since ι is simply the identity on $f_1(r), f_2(r)$.

2.2. **Split Morphisms.** Split morphisms play a fundamental role in homological algebra and category theory, as they provide a framework for understanding how certain morphisms can be *inverted* in a *controlled way*. In homological algebra, split monomorphisms and split epimorphisms are crucial because they ensure the existence of additional structure, such as direct sum decompositions, which are key to many proofs and constructions.

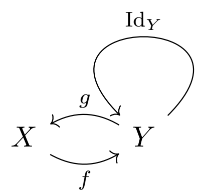
Definition 2.18. Let $\mathscr{C}$ be a category and let $f : X \to Y$ be a morphism.

(1) $f : X \to Y$ is a **split monomorphism** if f admits a left inverse $g : Y \to X$, meaning $g \circ f = \mathrm{Id}_X$.



The morphism g is called a **section** to f.

(2) $f : X \to Y$ is a **split epimorphism** if f admits a right inverse $g : Y \to X$, meaning $f \circ g = \mathrm{Id}_Y$.



The morphism g is called a **retract** to f.

The following theorem justifies the terminology and proves some additional results:

Lemma 2.19. *Let $\mathscr{C}$ be a category and let $f : X \to Y$ be a morphism.*

(1) *If f is a split monomorphism, then f is a monomorphism.*
(2) *If $\mathscr{C}$ is a concrete category and f is a split monomorphism, then f is injective.*
(3) *If $\mathscr{C} = \mathsf{Sets}$ and f is a monomorphism, then f is a split monomorphism.*

Dually,

(4) *If f is a split epimorphism, then f is an epimorphism.*
(5) *If $\mathscr{C}$ is a concrete category and f is a split epimorphism, then f is surjective.*
(6) *If $\mathscr{C} = \mathsf{Sets}$ and f is an epimorphism, then f is a split epimorphism.*

Proof. There exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$. For (1), let $h_1, h_2 : Z \to X$ be morphisms such that $f \circ h_1 = f \circ h_2$. Then

$$h_1 = \mathrm{Id}_X \circ h_1 = g \circ f \circ h_1 = g \circ \alpha \circ h_2 = \mathrm{Id}_X \circ h_2 = h_2,$$

implying that f is monic. For (2), let $x_1, x_2 \in X$ and assume that $f(x_1) = f(x_2)$. Then

$$x_1 = \mathrm{Id}_X(x_1) = g \circ f(x_1) = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2) = \mathrm{Id}_X(x_2) = x_2,$$

implying that f is injective. For (3), recall that a monomorphism is injective in Sets. Assume $x_0 \in X$[7]. Since f is injective, for each $y \in \operatorname{Im} f$ there exists a unique element

[7]Here we assume that $X \neq \emptyset$ since we invoke the axiom of choice.

$g(y) \in X$ such that $f(g(y)) = y$. This defines a function

$$\beta(y) = \begin{cases} \text{the unique } x \in X \text{ such that } f(x) = y, & \text{if } y \in \operatorname{Im} f, \\ x_0, & \text{if } y \notin \operatorname{Im} f. \end{cases}$$

We have $g \circ f = \mathrm{Id}_X$ by construction. Therefore, f is split monic.

(4) follows by a duality argument. (5) is clear. For (6), recall that an epimorphism is surjective in Sets. Thus, for each $y \in Y$, there exists $g(y) \in X$ such that $f(g(y)) = y$. This defines a function $g : Y \to X$, which satisfies $f \circ g = \mathrm{Id}_Y$. Therefore, f is a split epimorphism. □

According to Lemma 2.19, we have in a concrete category:

$$\text{Split Monomorphism} \implies \text{Injective} \implies \text{Monomorphism}$$
$$\text{Split Epimorphism} \implies \text{Surjective} \implies \text{Epimorphism.}$$

In Sets, Lemma 2.19 implies that the three notions coincide. Note that in an arbitrary category the middle term is no longer defined, but if it is deleted the resulting implication is still valid. Moreover, in an arbitrary category no two of these notions coincide in general as we now show.

Example 2.20. The following is a list of examples showing that the converse of the above implication is not true in general:

(1) Let $\mathscr{C} = \mathsf{Grp}$. The inclusion map $f : 2\mathbb{Z} \to \mathbb{Z}$ is an injective morphism. We claim that α is not a split monomorphism. Suppose to the contrary that there exists a morphism $g : \mathbb{Z} \to 2\mathbb{Z}$ such that $g \circ f = \mathrm{Id}_{2\mathbb{Z}}$. Then
$$2g(1) = g(2) = g(f(2)) = 2$$
so that $g(1) = 1$, contradicting that g maps into $2\mathbb{Z}$. Therefore, f is injective but not a split monomorphism.

(2) $\mathscr{C} = \mathsf{Top}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map, where the domain has the discrete topology and the codomain has the usual topology. Then α is surjective. Suppose that there exists a morphism $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = \mathrm{Id}_{\mathbb{R}}$. Since $f = \mathrm{Id}_{\mathbb{R}}$, we have $g = \mathrm{Id}_{\mathbb{R}}$.However, the set $\{0\}$ is open in $\mathbb{R}$ with the discrete topology, but its inverse image under g, which is also $\{0\}$, is not open in $\mathbb{R}$ with the usual topology. This contradicts that g is continuous. We conclude that f is injective but not a split monomorphism.

We have observed that an epimorphism generalizes the notion of an injective function to an arbitrary category. Similarly, a monomorphism generalizes the notion of a surjective function to an arbitrary category. Based on this, we can define the following:

Definition 2.21. Let $\mathscr{C}$ be a category and let $f : X \to Y$ be a morphism. f is a **bimorphism** if it is both a monomorphism and an epimorphism.

It is clear that a bimorphism generalizes the notion of an isomorphism in the sense that, in a concrete category, it is straightforward to verify that an isomorphism is a bijection. Furthermore, the previous results show that a bijection is a bimorphism.

The following characterization of isomorphism is useful.

Proposition 2.22. *Let $\mathscr{C}$ be a category and let $f : X \to Y$ be a morphism. Then f is an isomorphism if and only if f is both a split monomorphism and a split epimorphism.*

Proof. It follows immediately from the definitions that an isomorphism is both a split monomorphism and a split epimorphism. Conversely, $f : X \to Y$ be a morphism that is both a split monomorphism and a split epimorphism. So that there exist morphisms $g, g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ h = \mathrm{Id}_Y$. Then

$$f = f \circ \mathrm{Id}_Y = \beta \circ f \circ h = \mathrm{Id}_X \circ h = h,$$

Hence, f is an isomorphism. □

Remark 2.23. *In fact, we have the following slightly weaker claim:*

f is an isomorphism if and only if f is a split epimorphism and a monomorphism.

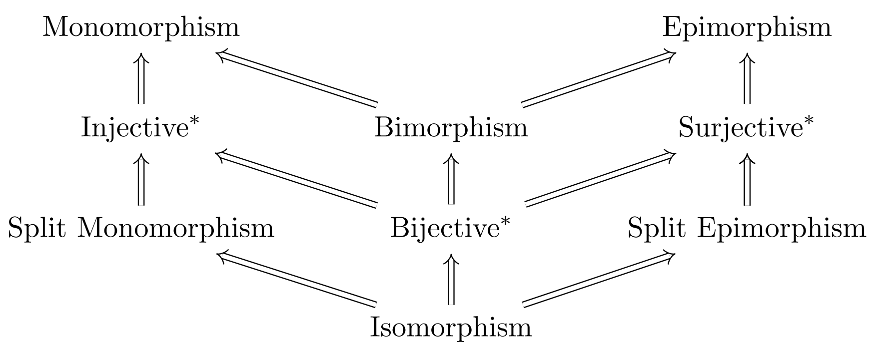
The forward direction is clear. We prove the backward direction. Since f is a split epimorphism, then there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$. We have:

$$f \circ g \circ f = \mathrm{Id}_Y \circ f = f \circ \mathrm{Id}_X \quad \textit{(definition of identities)}$$

Since f is a monomorphism, $g \circ f = \mathrm{Id}_X$. Hence, g is also a left inverse of f implying that f is an isomorphism. Dually, we have

f is an isomorphism if and only if f is a split monomorphism and an epimorphism.

The discussion above is summarized in the diagram below:



Remark 2.24. *The asterisk entries are to be included only if the category in question is concrete.*

We end this section with the following characterization of split monomorphisms and split epimorphisms:

Proposition 2.25. *Let $\mathscr{C}$ be a category and let $f : X \to Y$ be a morphism.*

(1) *f is a split epimorphism if for all $Z \in \mathscr{C}$, post-composition $f_* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is surjective.*

Dually,

(2) *f is a split monomorphism if for all $Z \in \mathscr{C}$, pre-composition $f^* : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is surjective.*

Proof. We prove (1). First assume that assume that f_* is surjective. Then for any $Z \in \mathscr{C}$ and $k \in \mathrm{Hom}(Z, Y)$, $k = f \circ g$, for some $g \in \mathrm{Hom}(Z, X)$. Now, suppose $Z = Y$ and $k = \mathrm{Id}_Y$, so we have that there exists a $g \in \mathrm{Hom}(Y, X)$ where $f \circ g = \mathrm{Id}_Y$. This implies that f is a split epimorphism. Now assume that f is a split epimorphism. There exists a morphism

$g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$. Let $Z \in \mathscr{C}$. Consider $f_* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$. Let $k \in \mathrm{Hom}(Z, Y)$. Consider $j = g \circ k$. Clearly, $j \in \mathrm{Hom}(Z, X)$. We have

$$f_*(g \circ k) = f \circ (g \circ k) = (f \circ g) \circ k = \mathrm{Id}_Y \circ k = k$$

This shows that f_* is surjective. (2) follows by a duality argument. □

3. Functors

A key principle in mathematics and category theory is that any mathematical object should be considered together with its accompanying notion of morphism. We can ask this about categories themselves: what are morphisms of categories? This question leads to the definition of a functor.

3.1. **Covariant Functors.**

Definition 3.1. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A **covariant functor** $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is a map $\mathscr{F} : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{D})$ such that for $X, Y \in \mathrm{Obj}(\mathscr{C})$, there is an induced map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(X), \mathscr{F}(Y)),$$

such that the following hold:

(1) $\mathscr{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{F}(X)}$ for each $X \in \mathrm{Obj}\,\mathscr{C}$
(2) If

$$A \xrightarrow{f} B \xrightarrow{g} C$$

then

$$\mathscr{F}(A) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(B) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(C)$$

That is

$$\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$$

Example 3.2. The following is a list of some *trivial* examples of functors:

(1) The identity functor is the functor $\mathrm{Id}_{\mathscr{C}}$ on any category $\mathscr{C}$ which leaves all objects and morphisms fixed.
(2) Let $\mathscr{C}'$ be a subcategory of $\mathscr{C}$. Then the inclusion $\mathscr{C}' \hookrightarrow \mathscr{C}$ is a covariant functor.

Example 3.3. A simple example is the power set functor $\mathscr{P} : \mathsf{Sets} \to \mathsf{Sets}$. $\mathscr{P}(X) = \mathcal{P}(X)$, the power set of X, and if $f \in \mathrm{Hom}(X, Y)$, then the map

$$\mathscr{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y),$$

which sends each $S \subseteq X$ to its image $f(S) \subseteq Y$. Since both $\mathscr{P}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{P}(X)}$ and $\mathscr{P}(g \circ f) = \mathscr{P}(g) \circ \mathscr{P}(f)$, this clearly defines a covariant functor.

Example 3.4. The following is a list of two related examples of functors:

(1) (**Forgetful Functor**) Each of the categories listed in Example 1.4 has a forgetful functor, a general term that is used for any functor that forgets structure and whose codomain is Sets. For example, functor from Vec_k to Sets, that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets.
(2) (**Free Functor**) Given any set S, one can build the free group $F(S)$ on S. This defines a free functor from Sets to Grp. Similarly, there exists free functors from Sets to $\mathsf{Ab}, \mathsf{CRing}, \mathsf{Vec}_k$, etc.

Example 3.5. The following is a list of two related examples of functors involving groups:

(1) Let G and H be groups regarded as one-object categories BG and BH. A functor $\mathscr{F} : \mathsf{BG} \to \mathsf{BH}$ must send the unique object of G to the unique object of H, so it is determined by its effect on morphisms which preserves composition. In other words, a functor $\mathscr{F} : \mathsf{BG} \to \mathsf{BH}$ is just a group homomorphism $G \to H$.

(2) A functor $\mathscr{F} : \mathsf{BG} \to \mathsf{Sets}$ consists of a set S together with, for each $g \in G$, a function $\mathscr{F}(g) : S \to S$, satisfying the functoriality axioms. Writing $(\mathscr{F}(g))(s) = g \cdot s$, we see that the functor $\mathscr{F}$ amounts to a set S together with a function

$$\begin{aligned} G \times S &\to S \\ (g, s) &\mapsto g \cdot s \end{aligned}$$

satisfying $(gh) \cdot s = g \cdot (h \cdot s)$ and $1_G \cdot s = s$ for all $g, h \in G$ and $s \in S$. Note that each $\mathscr{F}(g)$ is invertible with inverse $\mathscr{F}(g^{-1})$. In other words, a functor $\mathscr{F} : \mathsf{BG} \to \mathsf{Sets}$ is equivalent to a data of a group homomorphism $G \to \mathrm{Aut}(S)$ for some set S.

(3) Similarly, for any field k, a functor $\mathscr{F} : \mathsf{BG} \to \mathsf{Vec}_k$ is just a k-linear representation of G.

(4) Let $\mathsf{Conj} : \mathsf{Grp} \to \mathsf{Sets}$ represent the construction of the set of conjugacy classes of elements of a group, defined as follows:
   - For any group G, $\mathsf{Conj}(G) = \widehat{G}$, where $\widehat{G}$ is the set of conjugacy classes of G.
   - For any groups G and H and any group homomorphism $f : G \to H$, define the morphism

$$\mathsf{Conj}(f) : \widehat{G} \to \widehat{H} \qquad \mathsf{Conj}(f)([x]) = [f(x)] \text{ for each } [x] \in \widehat{G}$$

Clearly Conj is a functor. Since functors preserve isomorphisms, we clearly have that any pair of groups whose sets of conjugacy classes have different cardinalities (implying they are not isomorphic) cannot be isomorphic.

Example 3.6. For two pre-ordered sets $(S, \leq)$ and $(S', \leq)$, a covariant functor is given by an order-preserving map from $S \to S'$.

Example 3.7. The following is a list of some functors arising in homological algebra and algebraic topology:

(1) For each $n \in \mathbb{Z}$, there are functors

$$Z_n, B_n : \mathsf{Ch}_R \to \mathsf{Mod}_R$$

The functor Z_n computes the n-cycles and functor B_n computes the n-boundaries

$$\begin{aligned} Z_n C_\bullet &= \ker(d : C_n \to C_{n-1}) \\ B_n C_\bullet &= \mathrm{im}(d : C_{n+1} \to C_n) \end{aligned}$$

(2) If X is a topological space, singular homology in a given dimension n (where n is a natural number) is a covariant functor defined as:

$$H_n : \mathsf{Top} \to \mathsf{Ch}_{\mathbb{Z}} \to \mathsf{Ab}$$

First, each topological space X is assigned a singular chain complex such that $d \circ d = 0$ and then $H_n(X)$, the n-th homology group of X, is computed as follows:

$$H_n(X) = Z_n / B_n$$

It can be checked that H_n is functorial in the sense that each continuous map $f : X \to Y$ of spaces a corresponding homomorphism $H_n(f) : H_n(X) \to H_n(Y)$ of abelian groups.

(3) If X is a pointed topological space, the homotopy groups $\pi_n(X)$ of a space X can also be regarded as functors

$$\pi_n : \mathsf{Top}_* \to \mathsf{Grp}$$

since they depend on the choice of a base point in X.

(4) There are functors $\mathsf{Top} \to \mathsf{Htpy}$ and $\mathsf{Top}_* \to \mathsf{Htpy}_*$ that act as the identity on objects and send a (based) continuous function to its homotopy class.

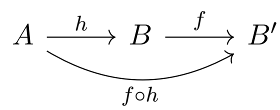
Example 3.8. If $\mathscr{C}$ is a locally small category and $A \in \mathscr{C}$, then the Hom functor $\mathrm{Hom}(A, -) : \mathscr{C} \to \mathsf{Sets}$, is defined by

$$\mathrm{Hom}(A,-)(B) = \mathrm{Hom}(A,B) \quad \text{for all } B \in \mathscr{C},$$

and if $f : B \to B'$ in C, then $\mathrm{Hom}(A,-)(f) : \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B')$ is given by

$$\mathrm{Hom}(A,-)(f) : h \mapsto f \circ h.$$

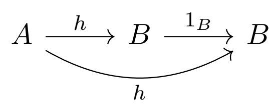
Note that the composite $f \circ h$ makes sense:



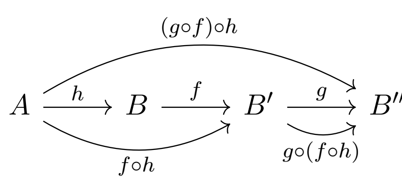
We call $\mathrm{Hom}(A,-)(f)$ the induced map, and we denote it by f_*; thus,

$$f_* : h \mapsto f \circ h.$$

If f is the identity map $1_B : B \to B$, then



Hence so that $(1_B)_* = 1_{\mathrm{Hom}(A,B)}$. Suppose now that $g : B' \to B''$. We have the following diagram:



Clearly,

$$g \circ (f \circ h) = (g \circ f) \circ h$$

Therefore, we have

$$(g \circ f)_* = g_* \circ f_*$$

Remark 3.9. *Recall that locally small means that for each A, B,* Hom(A,B) *is a set. This hypothesis is clearly necessary in order for the definition of the* Hom *functor above to make sense.*

3.2. **Contravariant Functors.**

Definition 3.10. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A **contravariant functor** $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is a map $\mathscr{F} : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$ such that for $X, Y \in \mathrm{Ob}(\mathscr{C})$, there is an induced map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(Y), \mathscr{F}(X)),$$

such that the following hold:

(1) $\mathscr{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{F}(X)}$ for each $X \in \mathrm{Obj}\,\mathscr{C}$
(2) If

$$A \xrightarrow{f} B \xrightarrow{g} C$$

then

$$\mathscr{F}(C) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(B) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(A)$$

That is

$$\mathscr{F}(g \circ f) = \mathscr{F}(f) \circ \mathscr{F}(g)$$

Remark 3.11. *When we say functor, we always mean a covariant functor. The reason for is that given a contravariant functor $\mathscr{C} \to \mathscr{D}$ is effectively a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$. Moreover, it is clear that functor from $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ can be identified with a functor $\mathscr{F} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$. Since $\mathscr{C} = (\mathscr{C}^{op})^{op}$ and $\mathscr{D} = (\mathscr{D}^{op})^{op}$, note that there is no difference between a functor $\mathscr{C}^{op} \to \mathscr{D}$ and a functor $\mathscr{C} \to \mathscr{D}^{op}$ as well. Hence, a contravariant functor can also be identified with a covariant functor $\mathscr{C} \to \mathscr{D}^{\mathrm{op}}$.*

Remark 3.12. *There is a functor from $\mathscr{C}$ to $\mathscr{C}^{\mathrm{op}}$ that maps every $X \in \mathscr{C}$ to $X \in \mathscr{C}^{\mathrm{op}}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ to the corresponding $f \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y)$. Call this functor ι. Any contravariant functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is equal to the composition of ι with the corresponding covariant functor from $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.*

Example 3.13. For two pre-ordered sets $(S, \leq)$ and $(S', \leq)$, a contravariant functor between the poset categories is given by an order-reversing map $S \to S'$.

Example 3.14. If $\mathscr{C}$ is a locally small category and $B \in \mathrm{obj}(\mathscr{C})$, then the contravariant Hom functor $\mathrm{Hom}(-, B) : \mathscr{C} \to \mathsf{Sets}$ for all $A \in \mathrm{obj}(C)$, by

$$\mathrm{Hom}(-, B)(A) = \mathrm{Hom}(A, B),$$

and if $f : A \to A'$ in $\mathscr{C}$, then $\mathrm{Hom}(-, B)(f) : \mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B)$ is given by

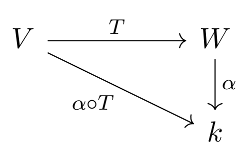
$$\mathrm{Hom}(-, B)(f) : h \mapsto h \circ f.$$

We call $\mathrm{Hom}(-, B)(f)$ the induced map, and we denote it by f^*; thus,

$$f^* : h \mapsto h \circ f.$$

An analysis similar to Example 3.8 shows that $\mathrm{Hom}(-, B)$ is a contravariant functor.

Example 3.15. The dual space functor from Vec_k to Vec_k is the contravariant functor of that sends a k-vector space V to its dual $V^* := \mathrm{Hom}(V, k)$ and a k-linear map $T : V \to W$ to its transpose $T^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ T$.



Example 3.16. Many constructions in modern mathematics are examples of contravariant functors. Below, we list some important constructions from various disciplines of mathematics:

(1) We have a contravariant functor from Top to CRing that sends a topological space X to the ring $C(X)$ of continuous complex-valued functions on X and a continuous map $f : X \to Y$ to the morphism[8]

$$\begin{aligned} f^* : C(Y) &\to C(X) \\ g &\mapsto g \circ f \end{aligned}$$

(2) The functor

$$\mathscr{O} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Poset}$$

that carries a topolgical space X to its poset category of open subsets, $\mathscr{O}(X)$, is a contravariant functor. A continuous map $f : X \to Y$ gives rise to a function $f^{-1} : \mathscr{O}(Y) \to \mathscr{O}(X)$ that carries an open subset $U \subseteq Y$ to its preimage $f^{-1}(U)$, which is open in X. This is the functorial definition of continuity.

(3) For a generic small category $\mathscr{C}$, a functor $\mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ is called a (set-valued) presheaf on $\mathscr{C}$. A typical example is the functor

$$\mathscr{F} : \mathscr{O}(X)^{\mathrm{op}} \to \mathsf{Sets}$$

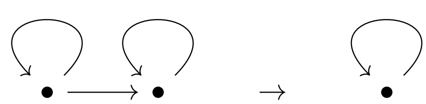
whose domain is the poset category $\mathscr{O}(X)$ of open subsets of a topological space X and whose value at $U \subseteq X$ is the set of continuous real-valued functions on U. The action on morphisms is by restriction: whenever $U \subseteq U'$ are open subsets of X, the map $\mathscr{F}(U') \to \mathscr{F}(U)$ is given by restriction. In fact, the presheaf given above is actually a CRing-valued presheaf[9].

Remark 3.17. *Here is a simple observation: a functor preserves isomorphisms. Consider an isomorphism $f : X \to Y$ in $\mathscr{C}$ with inverse $g : Y \to X$. Applying the functoriality axioms:*

$$\mathscr{F}(g) \circ \mathscr{F}(f) = \mathscr{F}(g \circ f)$$
$$\mathscr{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{F}(X)}\,.$$

Thus, $\mathscr{F}(g)$ is a left inverse to $\mathscr{F}(f)$. Exchanging the roles of f and g shows that $\mathscr{F}(g)$ is also a right inverse.

Remark 3.18. *We have seen above that a functor preserves isomorphisms. However, a functor need not reflect isomorphisms. That is, a functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ could be such that $\mathscr{F}(f)$ is an isomorphism in $\mathscr{D}$ even though but f is not an isomorphism in $\mathscr{C}$. Consider the obvious functor*



This functor doesn't reflect isomorphism since the the category on the left hand side is not a groupoid but the category on the right hand side is a groupoid.

[8]In fact, if we restrict it to the full subcategory of compact Hausdorff spaces, this functor lands in the subcategory of commutative C^*-algebras over $\mathbb{C}$. This functor is actually an equivalence of categories (to be defined later) and this is the content of the Gelfand-Naimark Theorem.

[9]In fact, it is a sheaf.

3.3. **Diagrams as Functors.** The intuitive notion of a diagram in a category $\mathscr{C}$—such as a chain of morphisms, a commutative square, or a more complex network of objects and arrows—can be captured precisely using functors. By selecting an appropriate indexing category that reflects the shape of the diagram, we can view a diagram as a structured assignment of objects and morphisms in $\mathscr{C}$, encoded via a functor.

Definition 3.19. A **diagram** in a category $\mathscr{C}$ is a functor $\mathscr{D} \to \mathscr{C}$, where $\mathscr{D}$ is a small category.

Example 3.20. The following are examples of diagrams in a category:

(1) A functor from the poset $\bullet \to \bullet$ to $\mathscr{C}$ is a choice of $A, B \in \mathscr{C}$ and a map $A \to B$.
(2) A functor from the poset $\mathbb{N}$ to $\mathscr{C}$ is the same as an infinite sequence in $\mathscr{C}$.
(3) A commutative square is the same as a functor out of the category:

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

into $\mathscr{C}$. A commutatice square in a category will then assume the following form: For instance, given objects and maps

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{i} & D \end{array}$$

We say that square commutes if $g \circ f = i \circ h$.

Generally, a diagram is said to commute if whenever there are two paths from an object X to an object Y, the map from X to Y obtained by composing along one path is equal to the map obtained by composing along the other.

Remark 3.21. *Here is a simple observation: functors preserve diagrams. A diagram in $\mathscr{C}$ is given by a functor $\mathscr{G} : \mathscr{D} \to \mathscr{C}$ where $\mathscr{D}$ is a small category. If $\mathscr{F} : \mathscr{C} \to \mathscr{H}$ is a functor, the composition $\mathscr{F} \circ \mathscr{G} : \mathscr{D} \to \mathscr{H}$ defines a diagram in $\mathscr{H}$.*

4. Natural Transformations

Category theory was invented by Saunders Mac-Lane and Samuel Eilenberg in the early 1940s, largely motivated by the desire to be precise about what is meant by (or should be meant by) a 'natural construction'. This leads us to the notion of a natural transformation. Another motivation for considering natural transformations is as follows. Having introduced functors as the natural notion of 'morphisms between categories,' the next natural step is to consider 'morphisms between functors.' Once we have the definition of a natural transformation, we will be able to formalize this intuition using the definition of the so-called functor categories (defined below).

Definition 4.1. Let $\mathscr{F}, \mathscr{G} : \mathscr{C} \to \mathscr{D}$ be functors. A **natural transformation** η from $\mathscr{F}$ to $\mathscr{G}$ is a collection of maps $\eta_X : \mathscr{F}(X) \to \mathscr{G}(X)$ for every $X \in \mathscr{C}$ such that for every morphism $f : X \to Y$ in $\mathscr{C}$, there's a commutative diagram:

$$\begin{array}{ccc} \mathscr{F}(X) & \xrightarrow{\mathscr{F}(f)} & \mathscr{F}(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ \mathscr{G}(X) & \xrightarrow[\mathscr{G}(f)]{} & \mathscr{G}(Y) \end{array}$$

A natural transformation η is a **natural isomorphism** if, for every $X \in \mathscr{C}$, the induced $\eta_X \in \mathrm{Hom}_{\mathscr{D}}(F(X), G(X))$ is an isomorphism.

Remark 4.2. *We denote a natural transformation, η, from $\mathscr{F}$ to $\mathscr{G}$ as:*

$$\mathscr{C} \; \begin{array}{c} \xrightarrow{\ \ \mathscr{F}\ \ } \\ \eta \Downarrow \\ \xrightarrow[\ \ \mathscr{G}\ \ ]{} \end{array} \; \mathscr{D}$$

The class of natural transformation between two functors $\mathscr{F}$ and $\mathscr{G}$ is denoted as $\mathrm{Nat}(\mathscr{F}, \mathscr{G})$*.*

Remark 4.3. *A natural transformation can be considered to be a 'morphism of functors.' This important observation leads to the notion of a 2-category.*

Example 4.4. The following are examples of natural transformations:

(1) Consider the covariant powerset functor $\mathscr{P} : \mathsf{Sets} \to \mathsf{Sets}$. For each set A, let $\eta_A : A \to \mathscr{P}(A)$ be the function $a \mapsto \{a\}$. Then η is a natural transformation between the power set functor and the identity functor.

(2) Let $\mathscr{C}$ be an arbitrary category and let $\mathscr{D} = \mathbb{Z}$. Functors $\mathscr{F}, \mathscr{G} : \mathbb{Z} \to \mathscr{C}$ are simply diagrams in $\mathscr{C}$ which can be visualized as follows:

$$\cdots \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \cdots$$

$$\cdots \longrightarrow G_{-1} \longrightarrow G_0 \longrightarrow G_1 \longrightarrow \cdots$$

A natural transformation from $\mathscr{F}$ to $\mathscr{G}$ is a sequence of maps $T_n : F_n \to G_n$ making the following diagram commute:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F_{-1} & \longrightarrow & F_0 & \longrightarrow & F_1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle T_{-1}} & & \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle T_1} & & \\ \cdots & \longrightarrow & G_{-1} & \longrightarrow & G_0 & \longrightarrow & G_1 & \longrightarrow & \cdots \end{array}$$

(3) For each $V \in \mathsf{Vect}_k$, let $V^{**} := \mathrm{Hom}(\mathrm{Hom}(V, k), k)$. The double dual space functor

$$\mathscr{F} : \mathsf{Vect}_k \to \mathsf{Vect}_k$$

is defined by sending $\mathscr{F}(V) = V^{**}$. The map $\mathrm{Ev} : V \to V^{**}$ that sends $v \in V$ to the linear function $\mathrm{Ev}_v : V^* \to k$ defines a natural transformation from the identity functor on Vect_k to the double dual space functor.

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{Ev}} & V^{**} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T^{**}} \\ W & \xrightarrow{\mathrm{Ev}} & W^{**} \end{array}$$

We check that the square commutes. By definition, $\mathrm{Ev}_{T(v)} : W^* \to k$ carries a functional $f : W \to k$ to $f(T(v))$. Moreover, we see that $T^{**}(\mathrm{Ev}_v) : W^* \to k$ carries a functional $f : W \to k$ to $f(T(v))$, which amounts to the same thing. In fact, this is a natural isomorphism if we restrict to the subcategory $\mathsf{Vec}_k^{\mathsf{Fd}}$. See Example 4.9(2) for a more general example.

(4) Recall that a functor from BG to Sets is a left G-set. Take two G-sets, S and T. Since S and T can be regarded as functors $\mathsf{BG} \to \mathsf{Sets}$, a natural transformation consists of a single map in Sets, $\alpha : S \to T$, such that

$$\alpha(g \cdot s) = g \cdot \alpha(s)$$

for all $s \in S$ and $g \in G$. In other words, it is just a map of G-sets, called a G-equivariant map.

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{\alpha} & T \end{array}$$

(5) Recall that a functor from BG to BH is a group homomorphism. Let η be a natural transformation between two functors $\mathscr{F}, \mathscr{H} : \mathsf{BG} \to \mathsf{BH}$ identified with group homomorphisms ϕ, ψ respectively. We have a diagram:

$$\begin{array}{ccc} *_H = \mathscr{F}(*_G) & \xrightarrow{\phi(g)} & \mathscr{F}(*_G) = *_H \\ {\scriptstyle \eta_{*G}}\downarrow & & \downarrow{\scriptstyle \eta_{*G}} \\ *_H = \mathscr{G}(*_G) & \xrightarrow[\psi(g)]{} & \mathscr{G}(*_G) = *_H \end{array}$$

Now η_{*_G} can be identified with an element $h \in H$. Hence, the above diagram reads

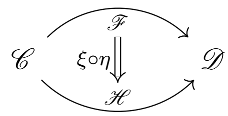
$$h = \cdot\phi(g) = \psi(g) \cdot h \iff \phi(g) = h^{-1} \cdot \psi(g) \cdot h$$

at the level of groups. Hence, η can be identified with a conjugacy between ϕ and ψ.

Natural transformations can be composed. Let $\mathscr{F}, \mathscr{G}, \mathscr{H} : \mathscr{C} \to \mathscr{D}$ be functors, and let η be a natural transformation between $\mathscr{F}$ and $\mathscr{G}$ and ξ a natural transformation between $\mathscr{G}$ and $\mathscr{H}$. We define $\xi \circ \eta$ a natural transformation from $\mathscr{F}$ to $\mathscr{H}$ by considering $\xi_X \circ \eta_X$ for each $X \in \mathscr{C}$.

$$\begin{array}{ccc} \mathscr{F}(X) & \xrightarrow{\mathscr{F}(f)} & \mathscr{F}(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ \mathscr{G}(X) & \xrightarrow[\mathscr{G}(f)]{} & \mathscr{G}(Y) \\ {\scriptstyle \xi_X}\downarrow & & \downarrow{\scriptstyle \xi_Y} \\ \mathscr{H}(X) & \xrightarrow[\mathscr{H}(f)]{} & \mathscr{H}(Y) \end{array} \qquad \text{with } \xi_X \circ \eta_X : \mathscr{F}(X) \to \mathscr{H}(X),\ \xi_Y \circ \eta_Y : \mathscr{F}(Y) \to \mathscr{H}(Y)$$

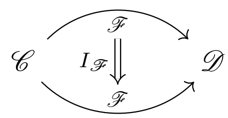
This natural transformation is denoted as:



There is also an identity natural transformation between a functor and itself by considering $\mathrm{Id}_{\mathscr{F}(X)}$ for each $X \in \mathscr{C}$.

$$\begin{array}{ccc} \mathscr{F}(X) & \xrightarrow{\mathscr{F}(f)} & \mathscr{F}(Y) \\ {\scriptstyle \mathrm{Id}_{\mathscr{F}(X)}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_{\mathscr{F}(Y)}} \\ \mathscr{F}(X) & \xrightarrow[\mathscr{F}(f)]{} & \mathscr{F}(Y) \end{array}$$

This natural transformation is denoted as:



For any two categories $\mathscr{C}$ and $\mathscr{D}$, there is a category whose objects are the functors from $\mathscr{C}$ to $\mathscr{D}$ and whose maps are the natural transformations between them. This is called the **functor category** from $\mathscr{C}$ to $\mathscr{D}$, written as $[\mathscr{C}, \mathscr{D}]$ or $\mathscr{D}^{\mathscr{C}}$. This formalizes the motivation remark made at the start of this subsection.

Example 4.5. The following are examples of functor categories:

(1) Let BG be a group. Then $[\mathsf{BG}, \mathsf{Sets}]$ is the category of left G-sets.
(2) Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be ordered sets viewed as categories. Given order-preserving maps $f, g : X \to Y$, viewed as functors, there is at most one natural transformation and there is one if and only if $f(x) \leq_Y g(x)$ for each $x \in X$. Hence, $[X, Y]$ is an ordered set too; its elements are the order-preserving maps from X to Y, and $f \leq g$ if and only if $f(x) \leq_Y g(x)$ for all $x \in X$.

What is the notion for which two categories are 'the same?' One might naively suggest two functors whose composition is the identity functor, but this is inadequate. The set of objects isn't very useful; it doesn't capture the structure of the category. In general, asking for equality of objects is worse than asking for isomorphism of objects. Here's the right notion of sameness.

Definition 4.6. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Then, a functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is an **equivalence of categories** if there's a functor $\mathscr{G} : \mathscr{D} \to \mathscr{C}$ such that there are natural isomorphisms such that $\mathscr{F} \circ \mathscr{G} \to \mathrm{Id}_{\mathscr{D}}$ and $\mathscr{G} \circ \mathscr{F} \to \mathrm{Id}_{\mathscr{C}}$.

Proposition 4.7. *A functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if and only if it is **fully faithful** (all the maps $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(\mathscr{F}(X), \mathscr{F}(Y))$ are isomorphisms) and **essentially surjective** (every $X \in \mathscr{D}$ is isomorphic to $\mathscr{F}(Z)$ for some $Z \in \mathscr{C}$).*

Proof. Assume $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories. Let $\mathscr{G} : \mathscr{D} \to \mathscr{C}$ be a functor such that $\mathscr{G} \circ \mathscr{F} \simeq I_{\mathscr{C}}$ and $\mathscr{F} \circ \mathscr{G} \simeq I_{\mathscr{D}}$ via the natural isomorphisms η and ξ. Let

$X, Y \in \mathscr{C}$. Consider the map $\beta : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(X), \mathscr{F}(Y))$. We show that β is an isomorphism by constructing its inverse explicitly. Consider the map

$$\alpha : \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(X), \mathscr{F}(Y)) \to \mathrm{Hom}_{\mathscr{C}}(\mathscr{G} \circ \mathscr{F}(X), \mathscr{G} \circ \mathscr{F}(Y)) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

The second map in the composition above - which we denote by γ -is well-defined since we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{G} \circ \mathscr{F}(X) & \xrightarrow{h} & \mathscr{G} \circ \mathscr{F}(Y) \\ {\scriptstyle \xi^{-1}(X)}\uparrow & & \downarrow{\scriptstyle \xi(Y)} \\ X & \xrightarrow[\gamma(h)]{} & Y \end{array}$$

$\alpha \circ \beta$ is the identity on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. Indeed, let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. The following diagram is commutative:

$$\begin{array}{ccc} \mathscr{G} \circ \mathscr{F}(X) & \xrightarrow{\mathscr{G} \circ \mathscr{F}(f)} & \mathscr{G} \circ \mathscr{F}(Y) \\ {\scriptstyle \xi(X)}\downarrow & & \downarrow{\scriptstyle \xi(Y)} \\ X & \xrightarrow[f]{} & Y \end{array}$$

This shows that $\xi(Y) \circ \mathscr{G} \circ \mathscr{F}(f) \circ \xi(X)^{-1} = f$, i.e., that $\alpha \circ \beta(f) = f$. In particular, the map β is injective, and the map α is surjective. Applying this result to $\mathscr{G}$, we see that the map α is also injective, hence it is bijective. Therefore, β is also bijective. Hence, α is fully faithful. Let $X \in \mathscr{D}$. Then $v : \mathscr{F}(\mathscr{G}(X)) \simeq X$ is an isomorphism, and $\mathscr{G}(X) \in \mathscr{C}$. This shows that $\mathscr{F}$ is essentially surjective.

Conversely, assume that $\mathscr{F}$ is a fully faithful and essentially surjective function. Since $\mathscr{F}$ is essentially surjective, for every object $X \in \mathscr{D}$ choose an object $X_{\mathscr{C}} \in \mathscr{C}$ such that $\mathscr{F}(Z_X)$ is isomorphic to X. Choose an isomorphism $\eta_X : X \to \mathscr{F}(\mathscr{G}(X))$. Define $\mathscr{G}$ on objects of $\mathscr{D}$ as follows:

$$\mathscr{G} : \mathscr{D} \to \mathscr{C} \qquad \mathscr{G}(X) = X_{\mathscr{C}}$$

If $X, Y \in \mathscr{D}$, and $f \in \mathrm{Hom}_{\mathscr{D}}(X, Y)$, then $\eta_Y \circ f \circ \eta_X^{-1}$ is a morphism $\mathscr{F}(\mathscr{G}(X)) \to \mathscr{F}(\mathscr{G}(Y))$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle \eta_X^{-1}} & & \downarrow{\scriptstyle \eta_Y} \\ \mathcal{F}(\mathscr{G}(X)) & \dashrightarrow & \mathcal{F}(\mathscr{G}(Y)) \end{array}$$

Since $\mathscr{F}$ is fully faithful, there is a unique morphism $\mathscr{G}(f) : \mathscr{G}(X) \to \mathscr{G}(Y)$ such that

$$\mathscr{F}(\mathscr{G}(f)) = \eta_Y \circ f \circ \eta_X^{-1}.$$

We check that $\mathscr{G}$ is indeed a functor. Let $f \in \mathrm{Hom}_{\mathscr{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{D}}(Y, Z)$. By definition of $\mathscr{G}$, we obtain

$$\begin{aligned} \mathscr{F}(\mathscr{G}(g \circ f)) &= \eta_Z \circ (g \circ f) \circ \eta_X^{-1} \\ &= (\eta_Z \circ g \circ \eta_Y^{-1}) \circ (\eta_Y \circ f \circ \eta_X^{-1}) \\ &= \mathscr{F}(\mathscr{G}(g)) \circ \mathscr{F}(\mathscr{G}(f)) \\ &= \mathscr{F}(\mathscr{G}(g) \circ \mathscr{G}(f)) \end{aligned}$$

Since $\mathscr{F}$ is fully faithful,

$$\mathscr{G}(g \circ f) = \mathscr{G}(g) \circ \mathscr{G}(f)$$

If $\mathrm{Id}_X \in \mathrm{Hom}_{\mathscr{D}}(X, X)$

$$\mathscr{F}(\mathscr{G}(\mathrm{Id}_X)) = \eta_X \circ \mathrm{Id}_X \circ \eta_X^{-1} = \mathrm{Id}_{\mathscr{F}(\mathscr{G}(X))} = \mathscr{F}(\mathrm{Id}_{\mathscr{G}(X)})$$

Since $\mathscr{F}$ is fully faitful,

$$\mathscr{G}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{G}(X)}$$

By construction, $\mathscr{F} \circ \mathscr{G}$ is naturally isomorphic to $\mathrm{Id}_{\mathscr{D}}$. We show $\mathscr{G} \circ \mathscr{F}$ is naturally isomorphic to $\mathrm{Id}_{\mathscr{C}}$. For any $X \in \mathscr{C}$, note that $\mathscr{F}(\mathscr{G}(\mathscr{F}(X)))$ is naturally isomorphic to $\mathscr{F}$, because, by definition of $\mathscr{F}$, we have an isomorphism

$$\eta_{\mathscr{F}(X)} : \mathscr{F}(X) \to \mathscr{F}(\mathscr{G}(\mathscr{F}(X)))$$

such that for any morphism $f : X \to Y$ in $\mathscr{C}$, we have

$$\mathscr{F}(f) \circ \eta_{\mathscr{F}(X)} = \eta_{\mathscr{F}(Y)} \circ \mathscr{F}(\mathscr{G}(\mathscr{F}(f)))$$

Since $\mathscr{F}$ is fully faithful, for each pair of objects $X, Y \in \mathscr{C}$ there is a unique morphism

$$\theta_X : X \to \mathscr{G}(\mathscr{F}(X)) \quad \text{such that} \quad \mathscr{F}(\theta_X) = \eta_{\mathscr{F}(X)}$$

We show θ is a natural transformation. Recall that

$$\mathscr{F}(f) \circ \eta_{\mathscr{F}(X)} = \eta_{\mathscr{F}(Y)} \circ \mathscr{F}(\mathscr{G}(\mathscr{F}(f)))$$

The definition of θ implies that

$$\mathscr{F}(f) \circ \mathscr{F}(\theta_X) = \mathscr{F}(\theta_Y) \circ \mathscr{F}(\mathscr{G}(\mathscr{F}(f))),$$

which implies by functionality implies that

$$\mathscr{F}(f \circ \theta_X) = \mathscr{F}(\theta_Y \circ \mathscr{G}(\mathscr{F}(f)))$$

Since $\mathscr{F}$ is fully faithful,

$$f \circ \theta_X = \theta_Y \circ \mathscr{G}(\mathscr{F}(f)),$$

which means precisely that θ is a natural transformation from the identity functor to $\mathscr{G} \circ \mathscr{F}$. As $\mathscr{F}(\theta_X)$ is an isomorphism for each X and $\mathscr{F}$ is fully faithful, θ_X is an isomorphism [10]. This shows that $\mathscr{G} \circ \mathscr{F}$ is naturally isomorphic to the identity functor on $\mathscr{C}$. □

Corollary 4.8. *Equivalence of categories is an equivalence relation.*

Proof. Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories. Clearly, $\mathscr{C} \simeq \mathscr{C}$ (reflexivity), and if $\mathscr{C} \simeq \mathscr{D}$, then $\mathscr{D} \simeq \mathscr{C}$ (symmetry) by definition. If $\mathscr{C} \simeq \mathscr{D}$ and $\mathscr{D} \simeq \mathscr{E}$, there exist functors $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ and $\mathscr{G} : \mathscr{D} \to \mathscr{E}$ that are fully faithful and essentially surjective. Since $\mathscr{G} \circ \mathscr{F}$ is also fully faithful and essentially surjective, it follows that $\mathscr{C} \simeq \mathscr{E}$ (transitivity). □

Example 4.9. Here are some examples of equivalence of categories:

(1) For any field k, the categories Mat_k and $\mathsf{Vec}_k^{\mathsf{Fd}}$ are equivalent. Define a functor as follows:

$$k^{(-)} : \mathsf{Mat}_k \to \mathsf{Vec}_k^{\mathsf{Fd}}$$

The functor $k^{(-)}$ sends n to the vector space k^n, equipped with the standard basis. Similarly, the functor $k^{(-)}$ send an $m \times n$-matrix, interpreted with respect to the standard bases on k^n and k^m, to the corresponding linear map $k^n \to k^m$. Clearly, $k^{(-)}$ is fully faithful. Since any finite-dimensional vector space admits a basis, it is

[10] We use the fact that if $\mathscr{F}$ is a fully faithful functor and $\mathscr{F}(f)$ is an isomorphism, then f is an isomorphism

isomorphic to k^n for some n. Hence, $k^{(-)}$ is essentially surjective. Hence, $k^{(-)}$ is an equivalence of categories:

$$\mathsf{Mat}_k \simeq \mathsf{Vec}_k^{\mathsf{Fd}}$$

(2) (Pontryagin Duality in $\mathsf{Ab}_{\mathsf{Fin}}$) Let $\mathscr{C} = \mathsf{Ab}_{\mathsf{Fin}}$. We show that $\mathscr{C}$ is equivalent to $\mathscr{C}^{\mathrm{op}}$. Define a functor

$$\mathscr{F} : \mathsf{Ab}_{\mathsf{Fin}} \to \mathsf{Ab}_{\mathsf{Fin}}^{\mathrm{op}}$$

that takes G to the dual group of G, given by $\widehat{G} = \mathrm{Hom}(G, \mathbb{S}^1)$. We first show that $G \simeq \widehat{G}$ as finite abelian groups. By the structure theorem for finite abelian groups, we have,

$$G = \bigoplus_{i=1}^{m} \mathbb{Z}_{p_i^{n_i}}$$

where p_i is prime and $m, n_1, \cdots, n_m \in \mathbb{N}$. Note that

$$\widehat{G} = \mathrm{Hom}\left(\bigoplus_{i=1}^{m} \mathbb{Z}_{p_i^{n_i}}, \mathbb{S}^1\right) \simeq \bigoplus_{i=1}^{m} \mathrm{Hom}(\mathbb{Z}_{p_i^{n_i}}, \mathbb{S}^1) = \bigoplus_{i=1}^{m} \widehat{\mathbb{Z}_{p_i^{n_i}}}.$$

Hence, it suffices to assume that $G \simeq \mathbb{Z}_{p_i^{n_i}}$. But $\mathrm{Hom}(\mathbb{Z}_{p_i^{n_i}}, \mathbb{S}^1)$ is the $p_i^{n_i}$-torsion in $\mathbb{S}^1$. This is the group set of $p_i^{n_i}$-th roots of unity in $\mathbb{C}$, which is to say roots of the polynomial $X^{p_i^{n_i}} - 1$ that is separable of degree $p_i^{n_i}$. There are $p_i^{n_i}$ such roots since $\mathbb{C}$ is algebraically closed. Clearly, this group is a finite abelian cyclic group. This shows that

$$\widehat{\mathbb{Z}_{p_i^{n_i}}} \simeq \mathbb{Z}_{p_i^{n_i}}$$

We can show that $\mathscr{F}$ is an equivalence of categories by arguing that $\mathscr{F} \circ \mathscr{F} \simeq \mathrm{Id}_{\mathsf{Ab}_{\mathsf{Fin}}}$. The map

$$\begin{aligned} \mathrm{Ev} : G &\to \widehat{\widehat{G}} \\ g &\mapsto \mathrm{Ev}_g, \end{aligned}$$

where $\mathrm{Ev}_g(\phi) = \phi(g)$ for all $\phi \in \widehat{G}$ defines a natural transformation from the identity functor on $\mathsf{Ab}_{\mathsf{Fin}}$ to $\mathscr{F} \circ \mathscr{F}$:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ev}} & \widehat{\widehat{G}} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T^{**}} \\ H & \xrightarrow{\mathrm{Ev}} & \widehat{\widehat{H}} \end{array}$$

We can verify that the square commutes as in the case of Vec_k. Thus, it remains to show that that Ev is an isomorphism. Clearly, Ev is a a group homomorphism. Since $|G| = |\widehat{G}|$ it suffices to show that Ev is injective. If $g \in G$ and $\phi(g) = 1$ for all $\phi \in \widehat{G}$, we claim $g = 1$. Equivalently, if $g \neq 1$, we construct some $\phi \in \widehat{G}$ such that $\phi(g) \neq 1$. Consider a decomposition

$$G = \bigoplus_{i=1}^{m} \mathbb{Z}_{p_i^{n_i}}$$

Write $g = (g_1, \cdots, g_m)$. Since $g \neq 1$, we have g_i for some $i = 1, \cdots, m$. WLOG, we can assume that $G = \mathbb{Z}_{p_i^{n_i}}$[11]. If $\mu_{p_i^{n_i}}(\mathbb{C})$ is cyclic group of $p_i^{n_i}$-th roots of unity in $\mathbb{C}$, we can choose an isomorphism of groups

$$\phi : G \simeq \mu_{p_i^{n_i}}(\mathbb{C})$$

This satisfies $\phi(g) \neq 1$ for all non-trivial $g \in \mathbb{Z}_{p_i^{n_i}}$.

Remark 4.10. *In general, $\mathscr{C}$ and $\mathscr{C}^{\text{op}}$ need not be equivalent. For instance, Pontryagin duality generally states that the category of discrete abelian groups, denoted by Ab, is dual to the category of compact Hausdorff abelian topological groups.*

We end this section by defining connected categories and the skeleton of a category, and by providing some basic results about them.

Definition 4.11. Let $\mathscr{C}$ be a category. $\mathscr{C}$ is **connected** if any pair of objects in $\mathscr{C}$ can be connected by a finite sequence of morphisms.

Proposition 4.12. *Let $\mathscr{C}$ be a category. If $\mathscr{C}$ is a connected groupoid, then $\mathscr{C} \simeq \mathsf{B}G$, where $G \simeq \mathrm{Hom}(X, X)$ for any $X \in X$.*

Proof. Choose any $X \in \mathscr{C}$ and let $G = \mathrm{Hom}(X, X)$. It is clear that G is a group. The inclusion $\mathsf{B}G \hookrightarrow \mathscr{C}$ mapping the unique object of $\mathsf{B}G$ to $X \in \mathscr{C}$ is fully faithful by definition, and essentially surjective since $\mathscr{C}$ is a connected groupoid. The claim follows from Proposition 4.7. □

Here is a sample corollary from homotopy theory:

Corollary 4.13. *Let X be a path-connected topological space. For $x, x' \in X$, we have $\pi_1(X, x) \simeq \pi_1(X, x')$.*

Proof. For any $x \in X$, we have

$$\mathrm{Hom}_{\Pi_1(X)}(x, x) = \pi_1(X, x)$$

Since $\Pi_1(X)$ is a connected groupoid, Proposition 4.12 implies

$$\mathsf{B}\pi_1(X, x) \simeq \Pi_1(X) \simeq \mathsf{B}\pi_1(X, x')$$

Hence, we have

$$\mathsf{B}\pi_1(X, x) \simeq \mathsf{B}\pi_1(X, x')$$

An equivalence between a 1-object categories is an isomorphism. By the characterization of functors between $\mathsf{B}\pi_1(X, x)$ and $\mathsf{B}\pi_1(X, x')$, we have that an isomorphism between $\mathsf{B}\pi_1(X, x)$ and $\mathsf{B}\pi_1(X, x')$ is a group homomorphism between $\pi_1(X, x)$ and $\pi_1(X, x')$. The claim follows. □

The previous results lead to the following definition:

Definition 4.14. Let $\mathscr{C}$ be a category. A **skeletal category** is one that contains exactly one object in each isomorphism class. The **skeleton of** $\mathscr{C}$ is the unique (up to isomorphism) skeletal category that is equivalent to $\mathscr{C}$.

Example 4.15. Here is a basic list of skeletons of some categories:

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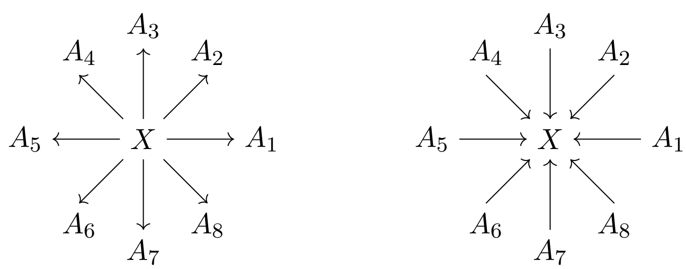
[11]Indeed, if there exists a $\psi : \mathbb{Z}_{p_i^{n_i}} \to \mathbb{S}^1$ such that $\psi(g_i) \neq 1$, so composing ψ with the projection $G \to \mathbb{Z}_{p_i^{n_i}}$ defines the desired ϕ.

(1) The skeleton of a connected groupoid is the group of automorphisms of any of its objects by Proposition 4.12.
(2) The skeleton of $\mathsf{Vec}_k^{\mathsf{Fd}}$ is Mat_k.
(3) The skeleton of the category $\mathsf{Sets}_{\mathsf{Fin}}$ is the category FinOrd whose objects are all sets of the form $\{0, \ldots, n-1\}$ for $n \in \mathbb{N}$, and whose morphisms are all functions between these sets.

5. Representable Functors

A key tenet of modern mathematics is the following meta-statement: a mathematical object is determined by its relationships to other objects. In other words, a mathematical object should be studied by considering the collection of maps to or from the object. What does it mean to study a mathematical object should be studied by considering the collection of maps to or from the object? Fortunately, we have already seen this idea before. Let $\mathscr{C}$ be a category. For each object $X \in \mathscr{C}$, the functors $\mathrm{Hom}(X, -)$ (Example 3.8) and $\mathrm{Hom}(-, X)$ (Example 3.14) define the 'vantage point' from and to an object $X \in \mathscr{C}$.



Recall that if $\mathscr{C}$ is a locally small category, each $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$ are sets for each $X \in \mathscr{C}$. This motivates the following definition:

Definition 5.1. Let $\mathscr{C}$ be a locally small category. The functor

$$\mathrm{Hom}(-, -) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Sets}$$

is called a a **two-sided represented functor**[12]

The above discussion suggests that the Hom functors are indeed quite useful in studying objects in a category. Other functors, however, may serve the same purpose in so far as they are equivalent to the Hom functors. This motivates the following definition:

Definition 5.2. Let $\mathscr{C}$ be a locally small functor. A covariant (or contravariant) functor $\mathscr{F}\mathscr{C}$: Sets is **representable** if there is a $X \in \mathscr{C}$ and a natural isomorphism between $\mathscr{F}$ and $\mathrm{Hom}(X, -)$ (or $\mathrm{Hom}(-, X)$).

Example 5.3. The following are examples of covariant representable functors:

(1) The identity functor $\mathrm{Id}_{\mathsf{Sets}} : \mathsf{Sets} \to \mathsf{Sets}$ is represented by a singleton set $\{*\}$. That is, for any set X, there is a natural isomorphism

$$\mathrm{Hom}(*, X) \simeq X$$

[12]A bifunctor.

that defines a bijection between elements $x \in X$ and functions $x : * \to X$ carrying the singleton element to x. Naturality says that for any $f : X \to Y$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(*, X) & \xrightarrow{f_*} & \mathrm{Hom}(*, Y) \\ \sim\downarrow & & \downarrow\sim \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes.

(2) The forgetful functor $\mathscr{F}_1 : \mathsf{Grp} \to \mathsf{Sets}$ is represented by the group $\mathbb{Z}$. That is, for any group G, there is a natural isomorphism

$$\mathrm{Hom}(\mathbb{Z}, G) \simeq \mathscr{F}_1(G)$$

that associates to every element $g \in G$, the unique homomorphism $\mathbb{Z} \to G$ that maps the integer 1 to g. This defines a bijection because every homomorphism $\mathbb{Z} \to G$ is determined by the image of the generator 1. Naturality says that for any group homomorphism $f : G \to H$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}, G) & \xrightarrow[f_*]{} & \mathrm{Hom}(\mathbb{Z}, H) \\ \sim\downarrow & & \downarrow\sim \\ \mathscr{F}_1(G) & \xrightarrow[f]{} & \mathscr{F}_1(H) \end{array}$$

commutes. Here $f : \mathscr{F}_1(G) \to \mathscr{F}_1(H)$ is the map between underlying sets.

(3) The n-fold forgetful functor $\mathscr{F}_n : \mathsf{Group} \to \mathsf{Sets}$ that sends G to $\prod_{i=1}^n G$ is represented by the free group on n generators. That is,

$$\mathrm{Hom}(*_{i=1}^n \mathbb{Z}, G) \simeq \mathscr{F}_n(G)$$

Naturality says that for any group homomorphism $f : G \to H$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}\left(*_{i=1}^n \mathbb{Z}, G\right) & \xrightarrow[f_*]{} & \mathrm{Hom}(*_{i=1}^n \mathbb{Z}, H) \\ \sim\downarrow & & \downarrow\sim \\ \mathscr{F}_n(G) & \xrightarrow[f]{} & \mathscr{F}_n(H) \end{array}$$

commutes. Here $f : \mathscr{F}_n(G) \to \mathscr{F}_n(H)$ is the map between underlying sets.

(4) The n-fold forgetful functor $\mathscr{G}_n : \mathsf{Ab} \to \mathsf{Sets}$ that sends an abelian group G to $\prod_{i=1}^n G$ is represented by is represented by the free abelian group on n generators. That is,

$$\mathrm{Hom}\left(\prod_{i=1}^n \mathbb{Z}, G\right) \simeq \mathscr{G}_n(G)$$

Naturality says that for any group homomorphism $f : G \to H$ of abelian groups, the diagram

$$\begin{array}{ccc} \mathrm{Hom}\left(\prod_{i=1}^n \mathbb{Z}, G\right) & \xrightarrow[f_*]{} & \mathrm{Hom}(\prod_{i=1}^n \mathbb{Z}, H) \\ \sim\downarrow & & \downarrow\sim \\ \mathscr{G}_n(G) & \xrightarrow[f]{} & \mathscr{G}_n(H) \end{array}$$

commutes. Here $f : \mathscr{G}_n(G) \to \mathscr{G}_n(H)$ is the map between underlying sets.

(5) The forgetful functor $\mathscr{H} : \mathsf{Ring} \to \mathsf{Sets}$ is represented by the unital ring $\mathbb{Z}[x]$.

$$\operatorname{Hom}(\mathbb{Z}[x], R) \simeq \mathscr{H}(R)$$

The argument is the same as in (2). Naturality says that for any ring homomorphism $f : R \to S$, the diagram

$$\begin{array}{ccc} \operatorname{Hom}(\mathbb{Z}[x], R) & \xrightarrow[f_*]{} & \operatorname{Hom}(\mathbb{Z}[x], S) \\ \sim\downarrow & & \downarrow\sim \\ \mathscr{H}(R) & \xrightarrow[f]{} & \mathscr{H}(S) \end{array}$$

commutes. Here $f : \mathscr{H}(R) \to \mathscr{H}(S)$ is the map between underlying sets.

(6) The functor $\mathscr{H}^\times : \mathsf{Ring} \to \mathsf{Sets}$ that sends a unital ring to its set of units is represented by the ring $\mathbb{Z}[x, x^{-1}]$ of Laurent polynomials in one variable. That is,

$$\operatorname{Hom}(\mathbb{Z}[x, x^{-1}], R) \simeq R^\times$$

A ring homomorphism $\mathbb{Z}[x, x^{-1}] \to R$ is defined by sending x to any unit of R and is completely determined by this assignment. The crucial point is that there are no ring homomorphisms that carry x to a non-unit. Naturality says that for any ring homomorphism $f : R \to S$ of unital rings, the diagram

$$\begin{array}{ccc} \operatorname{Hom}(\mathbb{Z}[x, x^{-1}], R) & \xrightarrow[f_*]{} & \operatorname{Hom}(\mathbb{Z}[x, x^{-1}], S) \\ \sim\downarrow & & \downarrow\sim \\ R^\times & \xrightarrow[f]{} & S^\times \end{array}$$

commutes. Here $f : \mathscr{H}^\times(R) \to \mathscr{H}^\times(S)$ is the map between underlying sets.

Example 5.4. The following are examples of contravariant representable functors:

(1) The contravariant powerset functor $\mathscr{P} : \mathsf{Sets} \to \mathsf{Sets}$ is represented by the set $\Omega = \{\top, \bot\}$ with two elements. The natural isomorphism

$$\operatorname{Hom}(X, \Omega) \simeq \mathscr{P}(X)$$

is defined by the bijection that associates a function $X \to \Omega$ with the subset that is the preimage of $\top$. Naturality says that for any set function $f : X \to Y$, the diagram

$$\begin{array}{ccc} \operatorname{Hom}(Y, \Omega) & \xrightarrow[f^*]{} & \operatorname{Hom}(X, \Omega) \\ \sim\downarrow & & \downarrow\sim \\ \mathscr{P}(Y) & \xrightarrow[f^{-1}]{} & \mathscr{P}(X) \end{array}$$

commutes.

(2) The contravariant functor $\mathscr{O} : \mathsf{Top} \to \mathsf{Sets}$ that sends X to $\mathscr{O}(X)$ is represented by the Sierpinski space $\mathscr{S}$: the topological space with two points, one closed and one open. The natural bijection

$$\operatorname{Hom}(X, \mathscr{S}) \simeq \mathscr{O}(X)$$

associates a continuous function $X \to \mathscr{S}$ to the pre-image of the open point. Naturality says that for any continuous function $f : X \to Y$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(Y, \mathscr{S}) & \xrightarrow[f^*]{} & \mathrm{Hom}(X, \mathscr{S}) \\ \sim\downarrow & & \downarrow\sim \\ \mathscr{O}(Y) & \xrightarrow[f^{-1}]{} & \mathscr{O}(X) \end{array}$$

commutes.

(3) Let $A, B \in \mathsf{Sets}$. The functor $\mathrm{Hom}(- \times A, B) : \mathsf{Sets}^{\mathrm{op}} \to \mathsf{Sets}$ that sends a set X to the set of functions $X \times A \to B$ is represented by $\mathrm{Hom}(A, B)$. That is

$$\mathrm{Hom}(X \times A, B) \simeq \mathrm{Hom}(X, \mathrm{Hom}(A, B))$$

Naturality says that for function $f : X \to Y$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(Y, \mathrm{Hom}(A, B)) & \longrightarrow & \mathrm{Hom}(X, \mathrm{Hom}(A, B)) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Hom}(Y \times A, B) & \longrightarrow & \mathrm{Hom}(X \times A, B) \end{array}$$

commutes. This natural isomorphism is referred to as *currying*.

(4) Let $A \in \mathsf{Ab}$ and any $n \in \mathbb{N} \cup \{0\}$. Singular cohomology with coefficients in A defines a functor

$$H^n(-; A) : \mathsf{Htpy}^{\mathrm{op}} \to \mathsf{Ab}$$

Passing to underlying sets and restricting to a subcategory of 'nice' topological spaces, such as the CW complexes, the resulting functor

$$H^n(-; A) : \mathsf{Htpy}_{\mathrm{CW}}^{\mathrm{op}} \to \mathsf{Sets}$$

is represented by the Eilenberg–MacLane space $K(A, n)$. That is, for any CW complex X, homotopy classes of maps $X \to K(A, n)$ stand in bijection with elements of the nth singular cohomology group $H^n(X; A)$ of X with coefficients in A:

$$[X, K(A, n)] \simeq H^n(-; A)$$

6. Yoneda's Lemma

Recall the definition of a representable functor. We can ask the following questions:

(1) Is every functor $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ representable by a functor in the functor category $\mathsf{Sets}^{\mathscr{C}}$?

(2) If so, given a functor $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$, what data is needed to define a natural isomorphism $\mathrm{Hom}(X, -) \simeq \mathscr{F}$ for some $X \in \mathscr{C}$? More generally, what data is needed to define a natural transformation $\mathrm{Hom}(X, -) \Rightarrow \mathscr{F}$ for some $X \in \mathscr{C}$?

These questions are answered by Yoneda's lemma (Lemma 6.2). First let's look at an example:

Example 6.1. Let $\mathscr{C} = \mathsf{BG}$. Recall that a covariant functor $\mathscr{F} : \mathsf{BG} \to \mathsf{Sets}$ corresponds to a left G-set, $\mathscr{F}(*) = X$. Assuming there exists a $X \in \mathsf{BG}$ that defines a natural transformation

$$\mathrm{Hom}(X, -) \Rightarrow \mathscr{F},$$

we have to have that $X = *$ such that

$$G \simeq \mathrm{Hom}(*, *) \Rightarrow \mathscr{F}(*) = X$$

Note that G is a left G-set with the action of G given by by left multiplication. Recall that a natural transformation

$$G \simeq \mathrm{Hom}(*, *) \Rightarrow \mathscr{F}(*) = X$$

corresponds to a G-equivariant map $\varphi : G \to X$. If such a φ exists, equivariance demands that

$$\varphi(g \cdot h) = g \cdot \varphi(h)$$

for all $g, h \in G$. Taking $h = e_G$, we see that $\varphi(g) = g \cdot \varphi(e_G)$. In other words, the choice of $\varphi(e_G) \in X$ defines φ. Clearly, any choice of $\varphi(e_G) \in X$ is permitted because the left action of G on G is free. We have a bijection:

$$\{G\text{-equivariant maps } \varphi : G \to X\ \} \longleftrightarrow \{\text{Elements of } X \text{ given by } \varphi(e_G) = x \in X\}$$

Based on Example 6.1, we might conjecture the following:

Let $\mathscr{C}$ be a locally small category. Given a functor $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ and an object X in $\mathscr{C}$, natural transformations between $\mathrm{Hom}(X, -)$ and $\mathscr{F}$ are in bijection with elements of $\mathscr{F}(X)$.

This is indeed the case and is the content of Yoneda's Lemma:

Lemma 6.2. ***(Yoneda's Lemma)*** *Let $\mathscr{C}$ be a locally small category. For any covariant functor $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ and any object $X \in \mathscr{C}$, there is a bijection*

$$\mathrm{Nat}(\mathrm{Hom}(X, -), \mathscr{F}) \simeq \mathscr{F}(X)$$

that associates a natural transformation $\eta : \mathrm{Hom}(X, -) \Rightarrow \mathscr{F}$ to the element $\eta_X(\mathrm{Id}_X) \in \mathscr{F}(X)$. Moreover, this correspondence is natural in both X and $\mathscr{F}$.

Remark 6.3. *In the proof of Lemma 6.2, for brevity in labeling commutative diagrams we have denoted the functor $\mathrm{Hom}(X, -)$ by T_X for each $X \in \mathscr{C}$.*

Proof. Let $\eta : \mathrm{Hom}(X, -) \to \mathscr{F}$ be a natural transformation. For all $Y, Z \in \mathscr{C}$, $f \in \mathrm{Hom}(X, Y)$, the following diagram must commute:

$$\begin{array}{ccc} \mathrm{Hom}(X, Y) & \xrightarrow{\eta_Y} & \mathscr{F}(Y) \\ {\scriptstyle T_X(f)}\downarrow & & \downarrow{\scriptstyle \mathscr{F}(f)} \\ \mathrm{Hom}(X, Z) & \xrightarrow[\eta_Z]{} & \mathscr{F}(Z) \end{array}$$

In particular, for each $X \in \mathscr{C}$, we have $\eta_X : \mathrm{Hom}(X, X) \to \mathscr{F}(X)$ such that $\mathrm{Id}_X \in \mathrm{Hom}(X, X)$. Consider the map:

$$\mu : \mathrm{Nat}(\mathrm{Hom}(X, -), \mathscr{F}) \to \mathscr{F}(X) \qquad \mu(\eta) = \eta_X(\mathrm{Id}_X).$$

We claim that μ is bijective. For each $Y \in \mathscr{C}$ and $f \in \mathrm{Hom}(X, Y)$, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}(X, X) & \xrightarrow{\eta_X} & \mathscr{F}(X) \\ {\scriptstyle T_X(f)}\downarrow & & \downarrow{\scriptstyle \mathscr{F}(f)} \\ \mathrm{Hom}(X, Y) & \xrightarrow[\eta_Y]{} & \mathscr{F}(Y) \end{array}$$

which means that

$$\begin{CD} \mathrm{Id}_X @>{\eta_X}>> \eta_X(\mathrm{Id}_X) \\ @V{T_X(f)}VV @VV{\mathscr{F}(f)}V \\ f @>>{\eta_Y}> \eta_Y(f) \end{CD}$$

If $\sigma : \mathrm{Hom}(X, -) \to \mathscr{F}$ is another natural transformation, then the above commutative diagram as above implies that

$$\sigma_Y(f) = (\mathscr{F}(\phi))(\sigma_X(\mathrm{Id}_X)).$$

Hence, if $\sigma_X(\mathrm{Id}_X) = \eta_X(\mathrm{Id}_X)$, then $\sigma_Y = \tau_Y$ for all $Y \in \mathscr{C}$ and, hence, $\sigma = \eta$ Therefore, μ is an injection. To prove surjectivity, let $A \in \mathscr{F}(X)$. For $Y \in \mathscr{C}$ and $\psi \in \mathrm{Hom}(X, Y)$, define

$$\eta_Y(\psi) = (\mathscr{F}(\psi))(A) \in \mathscr{F}(Y)$$

We show that τ is a natural transformation. That is, for $Z \in \mathscr{C}$ if $\theta \in \mathrm{Hom}(Y, Z)$, then the following diagram commutes:

$$\begin{CD} \mathrm{Hom}(X, Y) @>{\eta_X}>> \mathscr{F}(Y) \\ @V{T_X(\theta)}VV @VV{\mathscr{F}(\theta)}V \\ \mathrm{Hom}(X, Z) @>>{\eta_Y}> \mathscr{F}(Z) \end{CD}$$

Going clockwise, we have:

$$(\mathscr{F}(\theta))\eta_X(\psi) = \mathscr{F}(\theta)(\mathscr{F}(\psi)(A))$$

Going counterclockwise, we have

$$\eta_Y(T_X(\theta)(\psi)) = \eta_Y(\theta \circ \psi) = \mathscr{F}(\theta \circ \psi)(A).$$

Since $\mathscr{F}$ is a functor, however, $\mathscr{F}(\theta \circ \psi) = \mathscr{F}\theta \circ \mathscr{F}(\psi)$. Hence, τ is a natural transformation. Clearly,

$$\mu(\tau) = \tau_A(1_A) = \mathscr{F}(1_A)(A) = A,$$

and so μ is surjective. We now sketch the proof of naturality. Naturality in the functor states that if $\mathscr{G} : \mathscr{C} \to \mathsf{Sets}$ is another covariant functor, given a natural transformation $\beta : \mathscr{F} \Rightarrow \mathscr{G}$ the diagram

$$\begin{CD} \mathrm{Nat}(\mathrm{Hom}(X, -), \mathscr{F}) @>{\mu_{\mathscr{F}}}>> \mathscr{F}(X) \\ @V{(\beta_X)_*}VV @VV{\beta_X}V \\ \mathrm{Nat}(\mathrm{Hom}(X, -), \mathscr{G})) @>{\mu_{\mathscr{G}}}>> \mathscr{G}(X) \end{CD}$$

commutes. By definition, $\mu_{\mathscr{G}}(\beta \circ \eta) = (\beta \circ \eta)_X(\mathrm{Id}_X)$, which is $\beta_X(\eta_X(\mathrm{Id}_X)) = \beta_X(\mu_{\mathscr{F}}(\eta))$. Here we have used the definition of the composition of morphisms in an appropriate functor category $\mathsf{Sets}^{\mathscr{C}}$. This shows the diagram above commutes. Naturality in the object asserts that given a morphism $f : X \to Y$ in $\mathscr{C}$, the diagram

$$\begin{CD} \mathrm{Nat}(\mathrm{Hom}(X, -), \mathscr{F}) @>{\mu_{\mathscr{F}}}>> \mathscr{F}(X) \\ @VVV @VV{\mathscr{F}(f)}V \\ \mathrm{Nat}(\mathrm{Hom}(Y, -), \mathscr{G})) @>{\mu_{\mathscr{G}}}>> \mathscr{F}(Y) \end{CD}$$

commutes. It can be checked that the diagram commutes. See [Rie17, Theorem 2.2.4]. □

Remark 6.4. *Corollary 6.5 does not dualize to classify natural transformations from an arbitrary set-valued functor to a representable functor. That is,*

$$\mathrm{Nat}(\mathscr{F}, \mathrm{Hom}(X, -)) \not\simeq \mathscr{F}(X)$$

Here is an example. Let $\mathscr{C} = \mathsf{Sets}, \mathscr{F} = \mathrm{Id}_{\mathsf{Sets}}$, *and* $X = \emptyset$. *Then* $\mathrm{Hom}(\emptyset, X) = \{\emptyset\}$ *for each* $X \in \mathsf{Sets}$. *There is a unique natural transformation* $\alpha : \mathrm{Id}_{\mathsf{Set}} \Rightarrow \mathrm{Hom}(\emptyset, -)$. *This is not in bijection with* $\mathrm{Id}_{\mathsf{Set}}(\emptyset) = \emptyset$.

Here is the correct dual version of Corollary 6.5:

Corollary 6.5. ***(Dual Yoneda's Lemma)*** *Let* $\mathscr{C}$ *be a locally small category. For any contravariant functor* $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ *and any object* $X \in \mathscr{C}$, *there is a bijection*

$$\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(-, X), \mathscr{F}) \simeq \mathscr{F}(X)$$

that associates a natural transformation $\eta : \mathrm{Hom}(-, X) \Rightarrow \mathscr{F}$ *to the element* $\eta_X(\mathrm{Id}_X) \in \mathscr{F}(X)$. *Moreover, this correspondence is natural in both* X *and* $\mathscr{F}$.

Proof. The contravariant functor $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ corresponds to a covariant functor $\mathscr{F} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$. Since $\mathscr{C}$ is locally small, $\mathscr{C}^{\mathrm{op}}$ is also locally small since

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

is a set for all $X, Y \in \mathscr{C}$. By Lemma 6.2 applied to $\mathscr{F} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$, gives us a natural bijection

$$\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, -), \mathscr{F}) \simeq \mathscr{F}(X)$$

The claim follows since $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, -) = \mathrm{Hom}_{\mathscr{C}}(-, X)$. □

Remark 6.6. *Let's revisit the statement of Lemma 6.2 and Corollary 6.5. In* Sets, *recall that we have*

$$\mathrm{Hom}(\{*\}, X) \simeq X$$

for $X \in \mathsf{Sets}$. *Using this as a mnemonic* $\mathrm{Hom}(Y, X)$ *for* $Y \in \mathsf{Sets}$ *correspond to* Y-parameterized family of elements of X. *Naturally, this perspective extends to any category, allowing us to restate Corollary 6.5 as follows:*

The $\mathrm{Hom}(-, X)$*-elements of* X *are simply the usual elements of* $\mathscr{F}(X)$.

This formulation serves as a useful mnemonic and reinforces the intuition that an object is determined by its collection of probes given by Hom functors.

Building on the discussion in the previous section, we attempt to understand an object, X, in a locally small category by studying the covariant functor $\mathrm{Hom}(X, -)$. The covariant functor $\mathrm{Hom}(X, -)$ lies in the functor category $\mathsf{Sets}^{\mathscr{C}}$. Therefore, we expect there is a functor

$$\mathscr{G} : \mathscr{C} \to \mathsf{Sets}^{\mathscr{C}}$$

that sends an object X to $\mathrm{Hom}(X, -)$ and a morphism $f : X \to Y$ to the natural transformation $f_* : \mathrm{Hom}(X, -) \to \mathrm{Hom}(Y, -)$ is a functor. The functor $\mathscr{G}$ can be seen as a *representation* of $\mathscr{C}$ in terms of known structures. The Yoneda embedding theorem (Corollary 6.7) states that $\mathscr{G}$ is a fully faithful embedding. Hence, this suggests that it suffices to study the functor category $\mathsf{Sets}^{\mathscr{C}}$ instead of studying the locally small category $\mathscr{C}$. This approach is akin to (and in fact generalizes) the common method of studying a, for example, a ring by investigating the modules over that ring[13].

Corollary 6.7. ***(Yoneda's Embedding)*** *Let $\mathscr{C}$ be a locally small category. The assignment*

$$\begin{aligned}\mathscr{G} : \mathscr{C} &\to \mathsf{Sets}^{\mathscr{C}},\\ X &\mapsto \mathrm{Hom}(X, -)\end{aligned}$$

defines a contravariant functor. Moreover, $\mathscr{G}$ is fully faithful.

Proof. Let $X, Y \in \mathscr{C}$ and let $f \in \mathrm{Hom}(X, Y)$. Checking that $\mathscr{G}$ is a contravariant functor amounts to checking that there is a natural transformation $\eta : \mathrm{Hom}(Y, -) \Rightarrow \mathrm{Hom}(X, -)$. For $Z \in \mathscr{C}$, define:

$$\begin{aligned}\eta_Z : \mathrm{Hom}(Y, Z) &\to \mathrm{Hom}(X, Z),\\ \varphi &\mapsto \varphi \circ f\end{aligned}$$

via the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{h\circ f} & \downarrow h \\ & & Z \end{array}$$

The fact that η defines a natural transformation amounts to show that for each $Z, Z' \in \mathscr{C}$ and $g : Z \to Z'$, the following diagram is commutes:

$$\begin{array}{ccc} \mathrm{Hom}(Y, Z) & \xrightarrow{\eta_Z} & \mathrm{Hom}(X, Z) \\ {\scriptstyle T_Y(g)}\downarrow & & \downarrow{\scriptstyle T_X(g)} \\ \mathrm{Hom}(Y, Z') & \xrightarrow[\eta_{Z'}]{} & \mathrm{Hom}(X, Z') \end{array}$$

For each $\varphi \in \mathrm{Hom}(Y, Z)$, it amounts to having

$$\begin{aligned} T_X(g) \circ \eta_Z(\varphi) &= g \circ (\varphi \circ f)\\ &= (g \circ \varphi) \circ f\\ &= \eta_{Z'} \circ T_X(g)(\varphi). \end{aligned}$$

which is clearly true. The funcoriality axioms are easy to check and this detail is omitted. Hence, $\mathscr{G}$ is a contravariant functor which corresponds to the covariant functor $\mathscr{G} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}^{\mathscr{C}}$. $\mathscr{G}$ is also a fully faithful functor since for for each objects $X, Y \in \mathscr{C}$, the isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(Y, X) \simeq \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) \simeq \mathrm{Nat}(\mathrm{Hom}(X, -), \mathrm{Hom}(Y, -))$$

follows from Lemma 6.2 by letting $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ be the $\mathrm{Hom}(Y, -)$ functor. □

Remark 6.8. *Let $\mathscr{C}$ be a locally small category. Corollary 6.5 implies that the assignment*

$$\begin{aligned}\mathscr{G} : \mathscr{C} &\to \mathsf{Sets}^{\mathscr{C}^{\mathrm{op}}},\\ X &\mapsto \mathrm{Hom}(-, X).\end{aligned}$$

defines a covariant functor such that $\mathscr{G}$ is fully faithful.

[13] The ring takes the place of the category $\mathscr{C}$, and the category of modules over the ring is a category of functors defined on $\mathscr{C}$.

Remark 6.9. *The pair $(X, \mathscr{F})$ in the statement of Lemma 6.2 define an object in the product category $\mathscr{C} \times \mathsf{Sets}^{\mathscr{C}}$. There is a bifunctor $\mathrm{Ev} : \mathscr{C} \times \mathsf{Sets}^{\mathscr{C}} \to \mathsf{Sets}$ that maps $(X, \mathscr{F})$ to the set $\mathscr{F}(X)$. The Ev functor defines the co-domain of map:*

$$\mu : \mathrm{Nat}(\mathrm{Hom}(X, -), \mathscr{F}) \to \mathscr{F}(X)$$

in the proof of Lemma 6.2. The definition of the domain of μ makes use of the contravariant functor:

$$\mathscr{G} : \mathscr{C} \to \mathsf{Sets}^{\mathscr{C}},$$
$$X \mapsto \mathrm{Hom}(X, -).$$

The domain of μ can be identified as being given by the composite

$$\mathscr{C} \times \mathsf{Sets}^{\mathscr{C}} \to (\mathsf{Sets}^{\mathscr{C}})^{\mathrm{op}} \times \mathsf{Sets}^{\mathscr{C}} \to \mathsf{Nat}(\mathscr{C} \to \mathsf{Sets})$$
$$(X, \mathscr{F}) \mapsto (\mathrm{Hom}(X, -), \mathscr{F}) \mapsto \mathrm{Nat}(\mathrm{Hom}(X, -), \mathscr{F})$$

Here $\mathsf{Nat}(\mathscr{C} \to \mathsf{Sets})$ is the collection of natural transformations between a pair of functors $\mathscr{F}, \mathscr{G} : \mathscr{C} \to \mathsf{Sets}$ might not be a set. Here is where the size issues arise. If $\mathscr{C}$ is a small category, then $\mathsf{Set}^{\mathscr{C}}$ is a locally small category and $\mathsf{Nat}(\mathscr{C} \to \mathsf{Sets})$ can be replaced by Sets. However, if $\mathscr{C}$ is only a locally small category, $\mathsf{Set}^{\mathscr{C}}$ need not be a locally small category.

We now discuss several consequences of Corollary 6.7. We first present a strengthening of Proposition 2.6.

Proposition 6.10. *Let $\mathscr{C}$ be a locally small category. The following are equivalent:*

1. *$f : X \to Y$ is an isomorphism in $\mathscr{C}$.*
2. *For all objects $Z \in \mathscr{C}$, post-composition with f defines a natural transformation $f_* : \mathrm{Hom}(Z, X) \Rightarrow \mathrm{Hom}(Z, Y)$.*
3. *For all objects $Z \in \mathscr{C}$, pre-composition with f defines a natural transformation $f^* : \mathrm{Hom}(Y, Z) \Rightarrow \mathrm{Hom}(X, Z)$.*

Proof. By Corollary 6.7 we have a fully faithful functors $\mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}^{\mathscr{C}}$ and $\mathscr{C} \to \mathsf{Sets}^{\mathscr{C}^{\mathrm{op}}}$. For $X, Y \in \mathscr{C}$, we have bijections:

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) \simeq \mathrm{Hom}_{\mathsf{Sets}^{\mathscr{C}}}(\mathrm{Hom}_{\mathscr{C}}(Y, -), \mathrm{Hom}_{\mathscr{C}}(X, -)),$$
$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \simeq \mathrm{Hom}_{\mathsf{Sets}^{\mathscr{C}^{\mathrm{op}}}}(\mathrm{Hom}_{\mathscr{C}}(-, X), \mathrm{Hom}_{\mathscr{C}}(-, Y)).$$

To see that (1) implies (2) and (3), note that if there exists an isomorphism $f : X \to Y$, then there exists at least one natural transformation $\mathrm{Hom}_{\mathscr{C}}(Y, -) \Rightarrow \mathrm{Hom}_{\mathscr{C}}(X, -)$ and $\mathrm{Hom}_{\mathscr{C}}(-, X) \Rightarrow \mathrm{Hom}_{\mathscr{C}}(-, Y)$. Since post-composition and pre-composition with an isomorphism create another isomorphism, all components of f_* and f^* are isomorphisms. Therefore, f_* and f^* are natural isomorphisms implying (2) and (3). Now, we show that (2) implies (1) and that (3) implies (1). It suffices to show that (2) implies (1). Recall that full and faithful functors create isomorphisms. Since we have that $\mathrm{Hom}_{\mathscr{C}}(Y, -)$ and $\mathrm{Hom}_{\mathscr{C}}(X, -)$ are isomorphic by f_*, we know that X and Y are also isomorphic by f, since f_* is the image of f. This shows (2) implies (1). □

We discuss another important corollary of Corollary 6.7 that allows us to determine when two objects in a locally small category are isomorphic from the vantage point of Hom-functors.

Corollary 6.11. *Let $\mathscr{C}$ be a locally small category and let $X, Y \in \mathscr{C}$. Then*

$$X \simeq Y \iff \mathrm{Hom}(X, -) \simeq \mathrm{Hom}(Y, -) \iff \mathrm{Hom}(-, X) \simeq \mathrm{Hom}(-, Y)$$

Proof. This follows at once from Corollary 6.7 and Remark 6.8. Or directly from Proposition 6.10. □

We conclude our discussion by observing that Corollary 6.7 is a vast generalization of Cayley's theorem. In group theory, we can study arbitrary finite groups by examining group homomorphisms from arbitrary finite groups to the symmetric group. This is the statement of Cayley's Theorem, which essentially asserts that group multiplication *shuffles group elements around.* This profound insight is encapsulated in Cayley's Theorem, a cornerstone result in group theory. We end this section by asserting that Corollary 6.7 serves as a far-reaching generalization of Cayley's theorem from group theory.

Corollary 6.12. ***(Cayley's Theorem)*** *Let G be a group. Then G isomorphic to a subgroup of a permutation group.*

Proof. Example 6.1 identifies the object in the image of the contravariant Yoneda embedding $\mathsf{BG} \hookrightarrow \mathsf{Set}^{\mathsf{BG}}$ as the left G-set G. Corollary 6.7 asserts that the only G-equivariant endomorphisms of the G-set G are those maps defined by left multiplication with a fixed element of G. In particular, any G-equivariant endomorphism of G must be an automorphism. In this way, Corollary 6.7 defines an isomorphism between G and the automorphism group of the G-set G, an object in $\mathsf{Sets}^{\mathsf{BG}}$. Composing this with the faithful forgetful functor $\mathsf{Sets}^{\mathsf{BG}} \to \mathsf{Sets}$, we obtain an isomorphism between G and a subgroup of the automorphism group S_G of the set G. □

7. Universal Properties

Universal properties are a cornerstone of category theory, providing a unifying framework for defining and understanding mathematical structures. They encapsulate the essence of a construction by specifying it uniquely up to isomorphism, through its relationships with other objects. This abstraction allows mathematicians to generalize concepts across diverse mathematical areas. There are several approaches to defining universal properties. In this section, we describe universal properties using representable functors and the Yoneda Lemma.

Definition 7.1. Let $\mathscr{C}$ be a locally small category. An (initial) **universal property** of $X \in \mathscr{C}$ is given by the following data:

(1) A $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ representable functor represented by $\mathrm{Hom}(X, -)$,
(2) A **universal element**, $A \in \mathscr{F}(X)$, such that for any $Y \in \mathscr{C}$ and $B \in \mathscr{F}(Y)$, there exists a unique morphism $f : X \to Y$ in $\mathscr{C}$ such that $B = \mathscr{F}(f)(A)$ in Sets.

Example 7.2. Let $\mathbb{Z}[x] \in \mathsf{Ring}$. The (initial) universal property of $\mathbb{Z}[x]$ is given by the following data:

(1) The forgetful functor $\mathscr{H} : \mathsf{Ring} \to \mathsf{Sets}$ represented by ring $\mathbb{Z}[x]$.
$$\mathrm{Hom}(\mathbb{Z}[x], R) \simeq \mathscr{H}(R)$$
(2) A universal element $x \in \mathscr{G}(\mathbb{Z}[x])$. This holds because any ring homomorphism $f : \mathbb{Z}[x] \to R$ is uniquely determined by evaluation at a specific element, namely $f(x)$. In particular, for any $r \in \mathscr{F}(R)$, we can define a unique f by $f(x) = r$.

Example 7.3. Let $V, W \in \mathsf{Vec}_k$. Consider $V \otimes W \in \mathsf{Vec}_k$. The (initial) universal property of $V \otimes W$ is given by the following data:

(1) The functor
$$\mathrm{Bilin}(V, W; -) : \mathsf{Vec}_k \to \mathsf{Sets}$$
that sends a vector space U to the set of k-bilinear maps $V \times W \to U$. It is well-known that $\mathrm{Bilin}(V, W; -)$ is a represented by $V \otimes W \in \mathsf{Vec}_k$. That is,
$$\mathrm{Hom}(V \otimes W, U) \simeq \mathrm{Bilin}(V, W; U).$$

(2) A universal element $\otimes \in \mathrm{Bilin}(V, W; V \otimes W)$. Here $\otimes$ is a bilinear map such that
$$\otimes : V \times W \to V \otimes_k W \qquad \otimes(v, w) = v \otimes w$$
This holds because for any bilinear map $f : V \times W \to U$, there exists a unique linear map $\overline{f} : V \otimes W \to U$ such that the following diagram commutes:
$$\begin{array}{ccc} V \times W & \xrightarrow{\ \otimes\ } & V \otimes W \\ & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & U \end{array}$$

The universal property of the tensor product of a pair of vector spaces allows one to derive useful properties about it without relying on any specific construction of the tensor product. Here is a sample proposition:

Proposition 7.4. *Let $V, W \in \mathsf{Vec}_k$. Then $V \otimes W \simeq W \otimes V$.*

Proof. First note that the we have a natural isomorphism
$$\mathrm{Bilin}(V, W; U) \simeq \mathrm{Bilin}(W, V; U)$$
sends a bilinear map $f : V \times W \to U$ to the bilinear map $f' : W \times V \to U$ defined by $f'(w, v) = f(v, w)$. As a result, we have
$$\mathrm{Hom}(V \otimes W, U) \simeq \mathrm{Bilin}(V, W; U) \simeq \mathrm{Bilin}(W, V; U) \simeq \mathrm{Hom}(W \otimes V, U)$$
for each $U \in \mathsf{Vec}_k$. By Corollary 6.11, we have $V \otimes W \simeq W \otimes V$. □

Definition 7.5. Let $\mathscr{C}$ be a locally small category. A (terminal) **universal property** of $X \in \mathscr{C}$ is given by the following data:

(1) A $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ representable functor represented by $\mathrm{Hom}(-, X)$,
(2) A **universal element**, $A \in \mathscr{F}(X)$, such that for any $Y \in \mathscr{C}$ and $B \in \mathscr{F}(Y)$, there exists a unique morphism $f : Y \to X$ in $\mathscr{C}$ such that $A = \mathscr{F}(f)(B)$ in Sets.

Example 7.6. Let $\mathscr{S} \in \mathsf{Top}$ denote the Sierpinski space. The (terminal) universal property of $\mathscr{S}$ is given by the following data:

(1) The contravariant functor $\mathscr{O} : \mathsf{Top} \to \mathsf{Sets}$ that sends X to $\mathscr{O}(X)$ is represented by the Sierpinski space $\mathscr{S}$:
$$\mathrm{Hom}(X, \mathscr{S}) \simeq \mathscr{O}(X)$$

(2) A universal element $* \in \mathscr{O}(\mathscr{S})$ identified with the singleton open set in $\mathscr{S}$. This holds because for any open set $U \in \mathscr{O}(X)$, there exists a unique continuous function $f : X \to \mathscr{S}$ such that $f^{-1}(*) = U$.

Part 2. **Universal Constructions**

We study some of the most important constructions in category theory: limits, colimits, and adjoint functors. Limits and colimits capture universal properties that generalize constructions like products, coproducts, pullbacks, and pushouts. They allow us to describe and classify objects in terms of their relationships within a diagram. We also introduce adjoint functors, which provide a powerful abstraction for many dualities and constructions in mathematics.

8. Limits

The concept of limits unifies many familiar constructions in mathematics. Furthermore, the notion of limits provides an alternative way to define (final) universal properties.

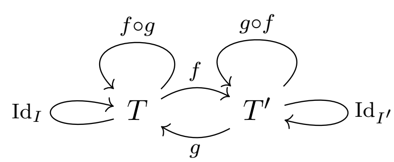
8.1. **Terminal Objects.** Terminal objects are a basic example of a limit, capturing the notion of a uniquely determined morphism from any object.

Definition 8.1. Let $\mathscr{C}$ be a category. A **terminal object** in $\mathscr{C}$ is an object $T \in \mathscr{C}$ that for all $X \in \mathscr{C}$, there exists a unique morphism $X \to T$.

It is easy to see that any two terminal objects are isomorphic.

Proposition 8.2. *Let $\mathscr{C}$ be a category. If T and T' are two terminal objects in $\mathscr{C}$, then $T \cong T'$.*

Proof. If T, T' are two initial objects, then there exist unique morphisms $f : T \to T'$ and $g : T' \to T$. Since T is a terminal object, the unique morphism $T \to T$ must be the identity. But $g \circ f$ is one such morphism. Therefore, $g \circ f = \mathrm{Id}_T$. Similarly, $f \circ g = \mathrm{Id}_{T'}$. Hence, $T \cong T'$.



This completes the proof. □

Example 8.3. The following are examples of terminal objects in various categories:

(1) Let $\mathscr{C} = \mathsf{Sets}$. The final object is a singleton, as any set can be uniquely mapped to a singleton. The unique map $S \to \{*\}$ is clear.
(2) Let $\mathscr{C} = \mathsf{Ab}$. The trivial abelian group is the final object. The unique map $A \to *$ is simply the zero map.
(3) Let $\mathscr{C} = \mathsf{Rings}$. The zero ring is the final object. The unique map $R \to 0$ is simply the zero map.

Let $\mathscr{C}$ be a category. Some mathematical objects in $\mathscr{C}$ with a (terminal) universal property can be seen as terminal objects in an appropriately defined category defined using the category $\mathscr{C}$.

Remark 8.4. *In what follows, we will not explicitly define these various 'derived' categories in which a mathematical object can be seen to be a terminal object. However, it should be clear from the context that there exists an underlying category in which this is true.*

As a warm-up to the general construction of limits, we consider two specific instances of limits: products and equalizers.

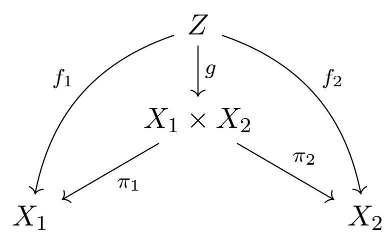
8.2. **Products.** Let $X_1, X_2 \in \mathsf{Sets}$. The Cartesian product given by

$$X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X, x_2 \in X_2\}$$

Note that a $(x_1, x_2) \in X_1 \times X_2$ can be identified with a function $\{*\} \to X_1 \times X_2$ or equivalently a pair of functions $\{*\} \to X_1$ and $\{*\} \to X_2$. More generally, $\{*\}$ can be replaced by any $Y \in \mathsf{Sets}$. The bijection between

$$\{\text{Functions } Y \to X_1 \times X_2\} \longleftrightarrow \{\text{Pairs of functions } Y \to X_1 \text{ and } A \to X_2\}$$

is given by composing with the canonical projection maps $\pi_{1,2} : X_1 \times X_2 \to X_{1,2}$. In fact, $X_1 \times X_2$ satisfies a universal property in the sense that given any functions $f_{1,2} : Z \to X_{1,2}$ there exists a unique function $g : Z \to X_1 \times X_2$ such that the following diagram commutes.



Even more generally, the two sets can be replaced by a collection of sets $\{X_i\}_{i \in I}$. This suggests the following definition.

Definition 8.5. Let $\mathscr{C}$ be a category, and let $(X_i)_{i \in I}$ a family of objects of $\mathscr{C}$. A **product** of $(X_i)_{i \in I}$, consists of an object $\prod_{i \in I} X_i$ and a family of morphisms

$$\prod_{i \in I} X_i \xrightarrow{p_i} X_i$$

for each $i \in I$ with the property that for $Y \in \mathscr{C}$ and families of morphisms $Y \xrightarrow{f_i} X_i$ for each $i \in I$ there exists a unique morphism $f : Y \to \prod_{i \in I} X_i$ such that $p_i \circ f = f_i$ for all $i \in I$.

Example 8.6. In Top, a family of objects $(X_i)_{i \in I}$ has a product. It is the set $\prod_{i \in I} X$ equipped with the product topology and the standard projection maps. The product topology is deliberately designed so that a function f

$$f : Y \to \prod_{i \in I} X,$$
$$y \mapsto (f_i(y))_{i \in I}.$$

is continuous if and only each f_i is a continuous function.

Example 8.7. Let $(S, \leq)$ be an ordered set. A lower bound for a family $(x_i)_{i \in I}$ of elements is an element $x \in S$ such that $x \leq x_i$ for all i. A greatest lower bound of the family is a lower bound greater than any other, denoted by $\bigwedge_{i \in I} x_i$. The greatest lower bounds serve as the products in $(X, \leq)$.

(1) In $(\mathbb{R}, \leq)$, the greatest lower bound of a family $(x_i)_{i \in I}$ is given by $\inf\{x_i \mid i \in I\}$ provided that the infimum exists.
(2) In the power set $(\mathcal{P}(S), \subseteq)$, the greatest lower bound of a family $(A_i)_{i \in I}$ is given $\bigcap_{i \in I} A_i$.

8.3. **Equalizers.** We now define equalizers. Equalizers capture the idea of selecting elements on which two morphisms agree. They are an important example of limits and arise frequently in categorical constructions.

Definition 8.8. Let $\mathscr{C}$ be a category and let $X \xrightarrow{f_1, f_2} Y$ be objects and morphisms in $\mathscr{C}$. An **equalizer** of f_1, f_2 is a $E \in \mathscr{C}$ together with a morphism $E \xrightarrow{i} X$ such that

$$E \xrightarrow{f} X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$$

is a diagram such that $f_1 \circ f = f_2 \circ f$ with the property that for any other diagram

$$A \xrightarrow{g} X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y,$$

such that $f_1 \circ g = f_2 \circ g$ there exists a unique map $\bar{f} : A \to E$ such that $f \circ \bar{f} = g$.

Remark 8.9. *We often draw the diagram of an equalizer as:*

$$\begin{array}{ccccc} A & \xrightarrow{g} & X & \underset{f_2}{\overset{f_1}{\rightrightarrows}} & Y \\ & \overset{\bar{f}}{\searrow} & \uparrow{\scriptstyle f} & & \\ & & E & & \end{array}$$

Example 8.10. Equalizers in Sets are easy to characterize. Consider sets and functions $f_1, f_2 : X \to Y$ and define

$$E = \{x \in X \mid f_1(x) = f_2(x)\},$$

with $f : E \to X$ as the inclusion map. One can verify that this indeed satisfies the definition of an equalizer.

Example 8.11. Let's see how the kernel is an instance of an equalizer in Mod_R. Let $M, N \in \mathsf{Mod}_R$ and let $f : M \to N$ be a R-module morphism. Recall that $\ker f$ is defined to be the following R-submodule of M:

$$\ker f = \{m \in M : f(m) = 0\}$$

If P is a R-module and $g : P \to M$ is an R-module morphism such such that $f \circ g = 0$, then there is a unique morphism $\overline{f} : P \to \ker f$ such that the following diagram commutes:

$$\begin{array}{ccccc} P & \xrightarrow{g} & M & \underset{0}{\overset{f}{\rightrightarrows}} & N \\ & \overset{\bar{f}}{\searrow} & \uparrow & & \\ & & \ker f & & \end{array}$$

Indeed, $f \circ g = 0$ implies that $g(P) \subseteq \ker f$. Therefore, $\overline{f}$ can be taken to be the map g whose codomain is restricted to $\ker f$. Since the diagram must commute, it is clear that this is the unique choice for $\overline{f}$. This shows that $\ker f$ is an equalizer.

8.4. **Cones.** We have now looked at two constructions: products and equalizers. These constructions are special instances of a more general construction known as a limit, which we now define. For binary products, the data required to define a binary product is a pair of objects

$$X \qquad Y$$

For equalizers, the data required to define an equalizer is a diagram

$$X \overset{f_1}{\underset{f_2}{\rightrightarrows}} Y$$

The data in both cases can be described by two functors with target category $\mathscr{C}$. Indeed, if $\mathscr{D}_1$ is the two-object discrete category,

$$\bullet \qquad \bullet$$

then the data of a binary product $X, Y \in \mathscr{C}$ can be identified with a functor $\mathscr{F} : \mathscr{D}_1 \to \mathscr{C}$. More generally if $\mathscr{D}_1 = \{*\}_{i \in I}$ is a discrete category with index set I, data of a product of $\{X_i\}_{i \in I} \in \mathscr{C}$ can be identified with a functor $\mathscr{F} : \mathscr{D}_1 \to \mathscr{C}$. If $\mathscr{D}_2$ is the two-object category with a single non-identity morphism between two different objects,

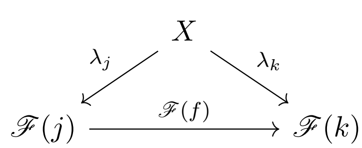
$$\bullet \longrightarrow \bullet$$

then the data of an equalizer can be identified with a functor $\mathscr{F} : \mathscr{D}_2 \to \mathscr{C}$. This motivates the following definition:

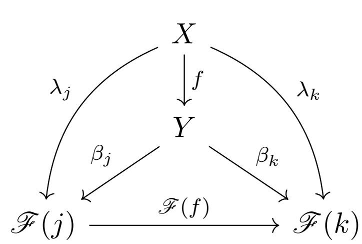
Definition 8.12. Let $\mathscr{C}$ be a and let $\mathscr{D}$ be a small category. A **cone on a diagram** $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ is defined by the following data:

(1) $X \in \mathscr{C}$ known as the summit of a cone,
(2) A natural transformation $\lambda : X \Rightarrow \mathscr{F}$ whose domain functor is the constant functor at X.

Explicitly, the data of a cone over a diagram $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ with summit $X \in \mathscr{C}$ is a collection of morphisms $\lambda_j : X \to \mathscr{F}(j)$, indexed by the objects $j \in J$ such that for each morphism $f : j \to k$ in $\mathscr{D}$, the following triangle commutes in $\mathscr{C}$:



Cones in $\mathscr{C}$ form a category denoted as $\mathrm{Cones}_{\mathscr{C}}$ defined in the natural way. In particular, morphisms in $\mathrm{Cones}_{\mathscr{C}}$ are morphisms $X \to Y$ in $\mathscr{C}$ such that the following diagram commutes:



Remark 8.13. *If $\mathscr{C}$ is a locally small category, then note that $\mathscr{C}^{\mathscr{D}}$ is a locally small category. This will guarantee that there is a set of cones with a fixed summit over the diagram. Most examples we discuss will fall in this case and are usually interested in this case.*

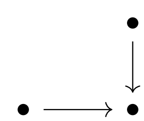
Definition 8.14. Let $\mathscr{C}$ be a, $\mathscr{D}$ be a small category and let $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. The **limit of a diagram** $\mathscr{F} : \mathscr{D} \to \mathscr{C}$, denoted $\varprojlim_{\mathscr{D}} \mathscr{F}$, is the final object in $\text{Cones}_{\mathscr{C}}$.

Remark 8.15. *By assuming from the outset that $\mathscr{D}$ is a small category, we are restricting ourselves to small limits.*

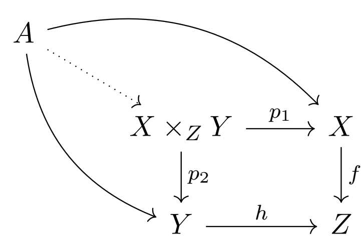
Example 8.16. If $\mathscr{D}$ is the empty category, then there is only one diagram of shape $\mathscr{D}$: the empty diagram (analogous to the empty function in set theory). A cone to the empty diagram in a category $\mathscr{C}$ is essentially just an object of $\mathscr{C}$. The limit of such a diagram is an object through which there exists a unique morphism from every other object in $\mathscr{C}$. This is precisely the definition of a terminal object.

Limits may or may not exist, but if they do, it is clear that they are unique up to isomorphism. The following is an important instance of a limit in a category.

Example 8.17. (**Fiber Products/Pullback**) Let $\mathscr{C}$ be a category and let $\mathscr{D}$ be the following category:



The limit of a diagram $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ is called a fiber product (or a pullback). A typical diagram representing a pullback in $\mathscr{C}$ is represented as follows:



We say that the diagram above represents the pullback of $X, Y \in \mathscr{C}$ along $Z \in \mathscr{C}$. The pullback object is denoted as $X \times_Z Y$. If $\mathscr{C} = \mathsf{Sets}$, we have

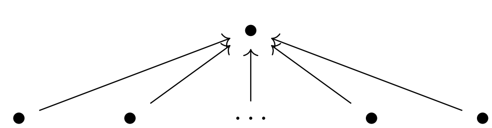
$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

with projections p_1 and p_2 given by the usual projection maps. If $X, Y \subseteq Z$ and f, g are the inclusion maps, note that we have

$$X \times_Z Y = X \cap Y$$

Hence, intersection of subsets provides an example of pullbacks.

Remark 8.18. *Let $\mathscr{C}$ be a category. More generally, one can define the pullback of a set of objects $\{X_i\}_{i \in I} \in \mathscr{C}$ along some $Z \in \mathscr{C}$. Here the corresponding diagram $\mathscr{D}$ assumes the form:*



Example 8.19. (**Inverse Limits**) Let $\mathscr{C}$ be a category and let $\mathscr{D} = (\mathbb{N}, \leq)^{\text{op}}$. A diagram $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ consists of objects and morphisms

$$\cdots \xrightarrow{s_3} X_2 \xrightarrow{s_2} X_1 \xrightarrow{s_1} X_0.$$

The limit is $\varprojlim_{i \in \mathbb{N}} X_i$. Such limits are sometimes referred to as inverse limits. For example, suppose that we have a set X_0 and a chain of subsets

$$\cdots \subseteq X_2 \subseteq X_1 \subseteq X_0.$$

The inclusion maps form a diagram in Set of the type above, and its limit is

$$\varprojlim_{i \in \mathbb{N}} X_i = \bigcap_{i \in \mathbb{N}} X_i$$

This example leads to the slogan that a limit is a *glorified intersection.*

8.5. **Existence of Limits.** A limit might not exist. Indeed, let $\mathscr{C} = \{\bullet_1, \bullet_2\}$ be the two object discrete category. It is clear that products don't exist in $\mathscr{C}$. When do limits exist? One has the following assertion, which we state without proof, that suffices for most purposes.

Proposition 8.20. *Let I be a set. Consider a diagram in* Sets *indexed by the set I. If the diagram is described by objects $\{X_i\}_{i \in I}$ and morphisms $f_{i,j} : X_i \to X_j$, the limit of the diagram is*

$$\left\{ (a_i)_{i \in I} \in \prod_i A_i \,\middle|\, f_{j,k}(a_j) = a_k \right\},$$

along with the obvious projection maps to each A_i.

Proof. Skipped. □

9. Colimits

The concept of colimits is formally dual to that of limits. Whereas limits capture terminal universal properties by means of cones into a diagram, colimits characterize initial universal properties via cocones out of a diagram.

9.1. **Initial Objects.** Initial objects are the dual notion to terminal objects, characterized by a unique morphism to every object in the category.

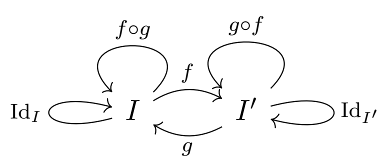
Definition 9.1. Let $\mathscr{C}$ be a category. An **initial object** in $\mathscr{C}$ is an object $T \in \mathscr{C}$ that for all $X \in \mathscr{C}$, there exists a unique morphism $I \to X$.

Remark 9.2. *An object that is initial and final is called a* ***zero object****.*

It is easy to see that any two initial objects are isomorphic.

Proposition 9.3. *Let $\mathscr{C}$ be a category. If I and I' are two initial objects in $\mathscr{C}$, then $I \cong I'$.*

Proof. If I, I' are two initial objects, then there exist unique morphisms $f : I \to I'$ and $g : I' \to I$. Since I is an initial object, the unique morphism $I \to I$ must be the identity. But $g \circ f$ is one such morphism. Therefore, $g \circ f = \text{Id}_I$. Similarly, $f \circ g = \text{Id}_{I'}$. Hence, $I \cong I'$.



This completes the proof. □

Example 9.4. The following are examples of initial objects in various categories:

(1) Let $\mathscr{C} = \mathsf{Sets}$. The initial object is the emptyset, $\emptyset$. The unique map $\emptyset \to S$ is clear.
(2) Let $\mathscr{C} = \mathsf{Ab}$. The trivial abelian group, $*$, is the initial object. The unique map $* \to A$ is simply the zero map.
(3) Let $\mathscr{C} = \mathsf{Rings}$. $\mathbb{Z}$ is the initial object, with the unique ring homomorphism $\mathbb{Z} \to R$ map determined by $1 \mapsto 1_R$.

Let $\mathscr{C}$ be a category. Some mathematical objects in $\mathscr{C}$ with an (initial) universal property can be see initial objects in an appropriately defined category defined using the category $\mathscr{C}$.

Remark 9.5. *In what follows, we will not explicitly define these various 'derived' categories in which a mathematical object can be seen to be an initial object. However, it should be clear from the context that there exists an underlying category in which this is true.*

As a warm-up to the general construction of limits, we consider two specific instances of limits: coproducts and coequalizers.

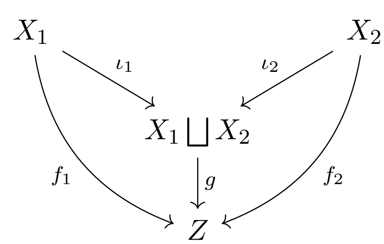
9.2. **Coproducts.** Let $X_1, X_2 \in \mathsf{Sets}$. The disjoint union given by

$$X_1 \bigsqcup X_2 = \{(x, i) \mid x \in X_i \text{ for } i = 1, 2\}$$

For $Y \in \mathsf{Sets}$, the bijection between

$$\{\text{Functions } X_1 \bigsqcup X_2 \to Y\} \longleftrightarrow \{\text{Pairs of functions } X_1 \to Y \text{ and } X_2 \to Y\}$$

is given by composing with the canonical inclusion maps $\iota_{1,2} : X_{1,2} \to X_1 \bigsqcup X_2$. In fact, $X_1 \bigsqcup X_2$ satisfies a universal property in the sense that, given any functions $f_1 : X_1 \to Z$ and $f_2 : X_2 \to Z$, there exists a unique function $g : X_1 \bigsqcup X_2 \to Z$ such that the following diagram commutes:



Even more generally, the two sets can be replaced by a collection of sets $\{X_i\}_{i \in I}$. This suggests the following definition.

Definition 9.6. Let $\mathscr{C}$ be a category and let I be a set and let $(X_i)_{i \in I}$ a family of objects of $\mathscr{C}$. A **coproduct** of $(X_i)_{i \in I}$, consists of an object $\coprod_{i \in I} X_i$ and a family of morphisms

$$X_i \xrightarrow{\iota_i} \coprod_{i \in I} X_i$$

for each $i \in I$ with the property that for $Y \in \mathscr{C}$ and families of morphisms $X_i \xrightarrow{f_i} Y$ for each $i \in I$ there exists a unique morphism $f : \coprod_{i \in I} X_i \to Y$ such that $f \circ \iota_i = f_i$ for all $i \in I$.

Example 9.7. In Top, a family of objects $(X_i)_{i\in I}$ has a coproduct. It is the set $\coprod_{i\in I} X$ equipped with the disjoint union topology and the standard inclusion maps. The disjoint union topology is deliberately designed so that a function f

$$f : \coprod_{i\in I} X \to Y, \quad f(x,i) \mapsto f_i(x),$$

is continuous if and only each f_i is a continuous function.

Example 9.8. Let $(S, \leq)$ be an ordered set. A upper bound for a family $(x_i)_{i\in I}$ of elements is an element $x \in S$ such that $x_i \leq x$ for all i. A smallest upper bound of the family is an upper bound lower than any other, denoted by $\bigvee_{i\in I} x_i$. The smallest upper bounds serves as the coproducts in $(X, \leq)$.

(1) In $(\mathbb{R}, \leq)$, the smallest upper bound of a family $(x_i)_{i\in I}$ is given by $\sup\{x_i \mid i \in I\}$ provided that the supremum exists.
(2) In the power set $(\mathcal{P}(S), \subseteq)$, the smallest upper bound of a family $(A_i)_{i\in I}$ is given by $\bigcup_{i\in I} A_i$.

9.3. **Coequalizers.** We now define coequalizers. Coequalizers are the dual notion to equalizers and capture the idea of identifying elements that are related by a pair of morphisms. They serve as a fundamental example of colimits in category theory.

Definition 9.9. Let $\mathscr{C}$ be a category and let $X \xrightarrow{f_1, f_2} Y$ be objects and morphisms in $\mathscr{C}$. An **coequalizer** of f_1, f_2 is a $C \in \mathscr{C}$ together with a morphism $Y \xrightarrow{f} C$ such that

$$X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \xrightarrow{f} C$$

is a diagram such that $f \circ f_1 = f \circ f_1$ with the property that for any other diagram

$$X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \xrightarrow{g} A$$

such that $g \circ f_1 = g \circ f_1$ there exists a unique map $\bar{f} : C \to A$ such that $\bar{f} \circ f = g$.

Remark 9.10. *We often draw the diagram of an coequalizer as:*

$$\begin{array}{ccccc} X & \underset{f_2}{\overset{f_1}{\rightrightarrows}} & Y & \xrightarrow{g} & A \\ & & \downarrow{\scriptstyle f} & \nearrow{\scriptstyle \bar{f}} & \\ & & C & & \end{array}$$

Example 9.11. Coequalizers in Sets are easy to characterize. Consider sets and functions $f_1, f_2 : X \to Y$ We must construct in some canonical way a set C and a function $f : Y \to C$ such that

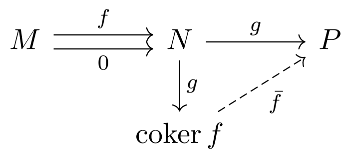
$$f(f_1(x)) = f(f_2(x))$$

for all $x \in X$. Let $\sim$ be the equivalence relation on Y generated by $f_1(x) \sim f_2(x)$ for all $x \in X$. Let $C = Y/\sim$ with $f : Y \to C$ as the quotient map. One can verify that this indeed satisfies the definition of a coequalizer.

Example 9.12. Let's see how the cokernel is an instance of an equalizer in Mod_R. Let $M, N \in \mathsf{Mod}_R$ and let $f : M \to N$ be a R-module morphism. Recall that $\operatorname{coker} f$ is defined to be the following R-module

$$\operatorname{coker} f = \frac{N}{\operatorname{im} f}$$

If P is a R-module and $g : N \to P$ is an R-module morphism such such that $g \circ f = 0$, then there is a unique morphism $\overline{f} : \operatorname{coker} f \to P$ such that the following diagram commutes:



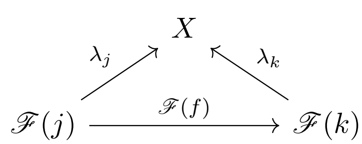
This shows that $\operatorname{coker} f$ is a coequalizer.

9.4. **Cocones.** We have now looked at two constructions: coproducts and coequalizers. These constructions are special instances of a more general construction known as a colimit, which we now define. Formally, expect the notion of a colimit to be dual to the notion of a limit. This motivates the following definition:

Definition 9.13. Let $\mathscr{C}$ be a and let $\mathscr{D}$ be a small category. A **cocone on a diagram** $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ is defined by the following data:

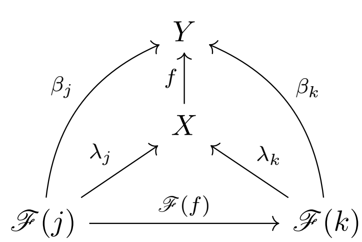
(1) $X \in \mathscr{C}$ known as the nadir of a cocone,
(2) A natural transformation $\lambda : \mathscr{F} \Rightarrow X$ whose target functor is the constant functor at X.

Explicitly, the data of a cocone over a diagram $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ with summit $X \in \mathscr{C}$ is a collection of morphisms $\lambda_j : X \to \mathscr{F}(j)$, indexed by the objects $j \in J$ such that for each morphism $f : j \to k$ in $\mathscr{D}$, the following triangle commutes in $\mathscr{C}$:



Remark 9.14. *Once again to guarantee that there is a set of cones with fixed summit over the diagram F, it suffices to assume that $\mathscr{D}$ is small and $\mathscr{C}$ is locally small, so that $\mathscr{C}^{\mathscr{D}}$ is locally small.*

Cocones in $\mathscr{C}$ form a category denoted as $\mathrm{Cocones}_{\mathscr{C}}$ defined in the natural way. In particular, morphisms in $\mathrm{Cocones}_{\mathscr{C}}$ are morphisms $X \to Y$ in $\mathscr{C}$ such that the following diagram commutes:



Remark 9.15. *If $\mathscr{C}$ is a locally small category, then note that $\mathscr{C}^{\mathscr{D}}$ is a locally small category. This will guarantee that there is a set of cones with a fixed summit over the diagram. Most examples we discuss will fall in this case and are usually interested in this case.*

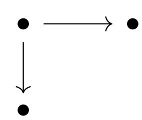
Definition 9.16. Let $\mathscr{C}$ be a, $\mathscr{D}$ be a small category and let $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. The **colimit of a diagram** $\mathscr{F} : \mathscr{D} \to \mathscr{C}$, denoted $\varinjlim_{\mathscr{D}} \mathscr{F}$, is the initial object in $\mathrm{Cocones}_{\mathscr{C}}$.

Remark 9.17. *By assuming from the outset that $\mathscr{D}$ is a small category, we are restricting ourselves to small colimits.*

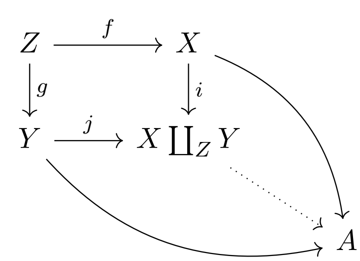
Example 9.18. If $\mathscr{D}$ is the empty category, then there is only one diagram of shape $\mathscr{D}$: the empty diagram. A cocone under the empty diagram in a category $\mathscr{C}$ is essentially just an object of $\mathscr{C}$. The colimit of such a diagram is an object that admits a unique morphism into every other object in $\mathscr{C}$. This is precisely the definition of an initial object.

Colimits may or may not exist, but if they do, it is clear that they are unique up to isomorphism. The following is an important instance of a colimit in a category.

Example 9.19. (**Pushout**) Let $\mathscr{C}$ be a category and let $\mathscr{D}$ be the following category:



The colimit of a diagram $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ is called a pushout. A typical diagram representing a pullback in $\mathscr{C}$ is represented as follows:



We say that the diagram above represents the pushout of $X, Y \in \mathscr{C}$ along $Z \in \mathscr{C}$. The pushout object is denoted as $X \coprod_Z Y$. If $\mathscr{C} = \mathsf{Sets}$, we have

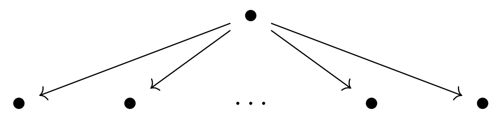
$$X \coprod_Z Y = (X \coprod Y)/\sim$$

where $\sim$ is the equivalence relation on $X \bigsqcup Y$ generated by $f(x) \sim g(z)$ for all $z \in Z$. The morphism $i : X \to X \bigsqcup_Z Y$ sends $x \in X$ to its equivalence class $[x]$ in $X \bigsqcup_Z Y$. Similarly, the morphism $j : Y \to X \bigsqcup_Z Y$ sends $y \in X$ to its equivalence class $[y]$ in $X \bigsqcup_Z Y$. If $Z = X \cap Y$ and f, g are inclusion maps, note that we have

$$X \bigsqcup_Z Y = X \bigcup Y$$

Hence, unions of subsets provides an example of pushouts.

Remark 9.20. *Let $\mathscr{C}$ be a category. More generally, one can define the pushout of a set of objects $\{X_i\}_{i\in I} \in \mathscr{C}$ along some $Z \in \mathscr{C}$. Here the corresponding diagram $\mathscr{D}$ assumes the form:*



Example 9.21. (**Direct Limits**) Let $\mathscr{C}$ be a category and let $\mathscr{D} = (\mathbb{N}, \leq)$. A diagram $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ consists of objects and morphisms

$$X_0 \xrightarrow{s_0} X_1 \xrightarrow{s_1} X_2 \xrightarrow{s_2} X_3 \cdots$$

The colimit is denoted as $\varinjlim_{i\in\mathbb{N}} X_i$. Such limits are sometimes referred to as inverse limits. For example, suppose that we have a set X_0 and a chain of subsets

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \cdots$$

The inclusion maps form a diagram in Sets of the type above, and its colimit is

$$\varinjlim_{i\in\mathbb{N}} X_i = \bigcup_{i\in\mathbb{N}} X_i$$

This example leads to the slogan that a limit is a *glorified union*.

9.5. **Existence of Colimits.** A colimit might not exist. Indeed, let $\mathscr{C} = \{\bullet_1, \bullet_2\}$ be the two object discrete category. It is clear that coproducts don't exist in $\mathscr{C}$. When do colimits exist? One has the following assertion, which we state without proof, that suffices for most purposes.

Proposition 9.22. *Let I be a filtered set*[14]*. Consider a diagram in* Sets *indexed by the set I. If the diagram is described by objects $\{X_i\}_{i\in I}$ and morphisms $f_{i,j} : X_i \to X_j$, the colimit of the diagram is*

$$\bigsqcup_{i\in I} A_i \Big/ \Big\{(a_i, i) \sim (a_j, j) \iff \textit{there exist } f_{i,j;k} : A_{i,j} \to A_k \ \textit{s.t}\ f_{i,k}(a_i) = f_{j,k}(a_j)\Big\}$$

Proof. Skipped. □

10. Adjoints

An adjunction between two categories formalizes the idea of two processes being 'inverse' to each other in a loose sense. The existence of adjoints often reflects deep structural properties and enables a systematic way to transfer information between categories.

Definition 10.1. Let $\mathscr{C}, \mathscr{D}$ be categories. Two functors $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ and $\mathscr{G} : \mathscr{D} \to \mathscr{C}$ are **adjoint pair of functors** if there is a bijection for all $A \in \mathscr{C}$ and $B \in \mathscr{D}$

$$\tau_{AB} : \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(A), B) \to \mathrm{Hom}_{A}(A, \mathscr{G}(B)).$$

[14]A filtered set is a setwith a partial order $\leq$ such that for any $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(A'), B) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(A), B) \\ \downarrow{\scriptstyle \tau_{A',B}} & & \downarrow{\scriptstyle \tau_{AB}} \\ \mathrm{Hom}_{\mathscr{D}}(A', \mathscr{G}(B)) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(A, \mathscr{G}(B)) \end{array}$$

We say that $(\mathscr{F}, \mathscr{G})$ forms an adjoint pair, and that $\mathscr{F}$ is **left-adjoint** to $\mathscr{G}$ and $\mathscr{G}$ is **right-adjoint** to $\mathscr{F}$.

Example 10.2. There's a pair of functors

$$\mathrm{Free} : \mathsf{Sets} \to \mathsf{Ab},$$
$$\mathrm{Forget} : \mathsf{Ab} \to \mathsf{Sets}.$$

These functors are adjoint pair. That is if G is a abelian group and S is a set, then

$$\mathrm{Hom}_{\mathsf{Ab}}(\mathrm{Free}(S), G) = \mathrm{Hom}_{\mathsf{Sets}}(S, \mathrm{For}(G))$$

This isomorphism reflects the fact that a group homomorphism from a free abelian group $\mathrm{Free}(S)$ to an abelian group G is uniquely determined by the image of the basis elements—that is, by the image of the set S under a function into the underlying set of G. This universal property of the free abelian group characterizes Free as the left adjoint to Forget.

Example 10.3. Let R be commutative ring. If M is a R-module, we have a functor

$$\mathrm{Hom}_R(M, -) : {}_R\mathsf{Mod} \to {}_R\mathsf{Mod}$$

Does it admit a left adjoint? Consider the set $\mathrm{Hom}(N, \mathrm{Hom}_R(M, P))$, for R-modules N, P. The elements of this set correspond to functions that assign to each $n \in N$ an R-linear map from $M \to P$. Such functions can be equivalently viewed as maps from the Cartesian product $M \times N$ to P that are R-bilinear (see Example 5.4(3)); that is, they are R-linear in each argument separately. This structure is reminiscent of the universal property of the tensor product, which asserts that R-bilinear maps $M \times N \to P$ correspond bijectively to R-linear maps $M \otimes_R N \to P$. Hence, we obtain a natural isomorphism

$$\mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P)) \cong \mathrm{Hom}_R(M \otimes_R N, P),$$

This exhibits the classical tensor-Hom adjunction: the functor $M \otimes_R -$ is left adjoint to $\mathrm{Hom}_R(M, -)$.

Given an adjoint pair of functors, one can often exploit properties of one functor to deduce corresponding properties of the other. This can be viewed as a categorical analogue of classical calculus, where we are interested in understanding when certain operations—such as limits, integrals, or derivatives—can be interchanged. In the categorical setting, we aim to determine when (co)limits can be commuted with functors, and how such behavior relates to the adjointness of those functors. A number of results in this direction are based on the following result, which, roughly speaking, states that the Hom functors commutes with (co)limits.

Proposition 10.4. ***(Hom Commutes with (Co)limits)*** *Let $\mathscr{C}$ be a locally small category, and let $X \in \mathscr{C}$. Let $\{A_i\}_{i \in I}$ be a collection of objects in $\mathscr{C}$ such that $\varprojlim_I A_i$ and $\varinjlim_I A_i$ exist.*

(1) *Let $X \in \mathscr{C}$. The covariant* Hom *functor (Example 3.8) commutes with limits:*

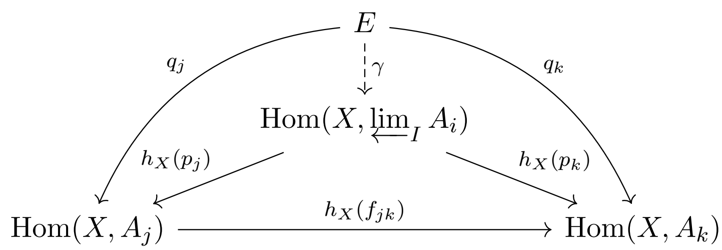
$$\varprojlim_I(\mathrm{Hom}(X, A_i)) = \mathrm{Hom}(X, \varprojlim_I A_i)$$

(2) *Let $X \in \mathscr{C}$. the contravariant* Hom *functor (Example 3.14) commutes with colimits:*

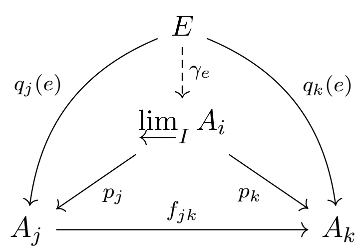
$$\mathrm{Hom}(\varinjlim_I A_i, X) = \varprojlim_I(\mathrm{Hom}(A_i, X))$$

Proof. The proof is given below:

(1) We write $\mathrm{Hom}(X, -)$ and $h_X(-)$ interchangeably for the covariant Hom functor. Let $f_{jk} : A_j \to A_k$ and $p_j : \varprojlim_I A_i \to A_j$ be the relevant morphisms defining $\varprojlim_I A_i$. Fix some $E \in \mathsf{Sets}$. It suffices to show that $\mathrm{Hom}(X, \varprojlim_I A_i)$ satisfies the universal property of $\varprojlim_I(\mathrm{Hom}(X, A_i))$ as shown in the following diagram:



For $e \in E$, note that $q_j(e) \in \mathrm{Hom}(X, A_j)$ and $q_k(e) \in \mathrm{Hom}(X, A_k)$. By the universal property of limits, there is a unique morphism $\gamma_e : E \to \varprojlim_I A_i$ such that the following diagram commutes:



This allows us to define the unique γ by $\gamma(e) = \gamma_e$. It is easy to see that the defined γ is uniquely defined due to the universal property of limits in $\mathscr{C}$.

(2) Note that $\mathrm{Hom}(-, X)$ is a covariant functor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{C}$. By (1), we have that:

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(\varprojlim_I A_i, X) = \varprojlim_I \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A_i, X)$$

Since limits in $\mathscr{C}^{\mathrm{op}}$ correspond to colimits in $\mathscr{C}$, we have that the following holds:

$$\mathrm{Hom}_{\mathscr{C}}(\varinjlim_I A_i, X) = \varprojlim_I \mathrm{Hom}_{\mathscr{C}}(A_i, X)$$

This completes the proof. ∎

We can use Proposition 10.4 to prove quite a few adjoint relationships. First, we can show that tensor products commute with colimits.

Proposition 10.5. *Let R be a ring and let I be a directed, filtered set. Additionally, let N be a left R-module and let $\{M_i\}_{i\in I}$ be right R-modules. Then:*

$$\varinjlim_I (M_i \otimes_R N) \cong (\varinjlim_I M_i) \otimes_R N$$

Proof. We exploit Example 10.3 and Proposition 10.4. For any R-module P, we have the following

$$\begin{aligned}
&\operatorname{Hom}(\varinjlim_I (M_i \otimes_R N), P) \\
&\quad\cong \varprojlim_I \operatorname{Hom}(M_i \otimes_R N, P) \\
&\quad\cong \varprojlim_I \operatorname{Hom}(N, \operatorname{Hom}(M_i, P)) \\
&\quad\cong \operatorname{Hom}(N, \varprojlim_I \operatorname{Hom}(M_i, P)) \\
&\quad\cong \operatorname{Hom}(N, \operatorname{Hom}(\varinjlim_I M_i, P)) \\
&\quad\cong \operatorname{Hom}((\varinjlim_I M_i) \otimes_R N, P)
\end{aligned}$$

Therefore:

$$\operatorname{Hom}(\varinjlim_I (M_i \otimes_R N), -) \cong \operatorname{Hom}((\varinjlim_I M_i) \otimes_R N, -).$$

By Corollary 6.11, we have:

$$\varinjlim_I (M_i \otimes_R N) \cong (\varinjlim_I M_i) \otimes_R N.$$

This completes the proof. □

Part 3. Abelian Categories

Many constructions and results in ${}_R\mathsf{Mod}$, the category of left R modules over a ring R, naturally extend to other contexts, such as finitely generated R-modules, graded R-modules over a graded ring, R, or sheaves of R-modules. To unify these settings and avoid repeating similar arguments, it is useful to adopt a general framework. This leads to the study of abelian categories which provide a robust framework for algebra and homological algebra.

11. Additive Categories

An additive category is, in particular, a pre-additive category. Therefore, we first consider the definition of a pre-additive category.

Definition 11.1. A category $\mathscr{C}$ is a **pre-additive** category if:

(1) For all $X, Y \in \mathscr{C}$, $\mathrm{Hom}(X, Y)$ is an abelian group.
(2) For all $X, Y, Z \in \mathscr{C}$, the composition

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \longrightarrow \mathrm{Hom}(X,Z)$$

is bilinear. That is,

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2,$$
$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$$

for $f_1, f_2 \in \mathrm{Hom}(X, Y)$ and $g_1, g_2 \in \mathrm{Hom}(Y, Z)$.

Remark 11.2. *If $\mathscr{C}$ is a pre-additive category, note that for any object $X \in \mathscr{C}$, the abelian group $\mathrm{Hom}(X, X)$ naturally acquires a ring structure. The multiplication is given by composition of morphisms.*

Example 11.3. Ab is a pre-additive category.

We now define an additive category by building upon the notion of a pre-additive category. Specifically, we augment the definition of a pre-additive category by requiring the existence of special types of objects.

Definition 11.4. An **additive** category is a pre-additive category $\mathscr{C}$ such that:

(1) $\mathscr{C}$ has an object, called the zero object, that is both an initial object and a terminal object,
(2) $\mathscr{C}$ has all finite products: given any two objects Y and Y in $\mathscr{C}$, there exists a product of X and Y in $\mathscr{C}$.

Remark 11.5. *In an additive category, $\mathrm{Hom}(X, Y) \neq 0$ for each $X, Y \in \mathscr{C}$. Indeed, if $0_{\mathscr{C}}$ is the zero object, there is always the zero morphism, 0_{XY}, between X and Y defined as the composition:*

$$\overbrace{X \longrightarrow 0_{\mathscr{C}} \longrightarrow Y}^{0_{XY}}$$

Note 0_{XY} is unique since the morphisms $X \to 0_{\mathscr{C}}$ and $Y \to 0_{\mathscr{C}}$ are unique. Note that the composite of a zero morphism, 0_{XY}, with an arbitrary morphism is again a zero morphism. In what follows, we use the notation 0_X to denote the morphism:

$$\overbrace{X \longrightarrow 0_{\mathscr{C}} \longrightarrow X}^{0_{XX}}$$

Remark 11.6. $0_{\mathscr{C}}$ *is uniquely determined by the condition:* $\mathrm{Id}_{0_{\mathscr{C}}} = 0_{0_{\mathscr{C}}} \in \mathrm{Hom}(0_{\mathscr{C}}, 0_{\mathscr{C}})$. *The necessary condition is clear. Conversely, if* $X \in \mathscr{C}$ *such that* $\mathrm{Id}_X = 0_X$*, then for any* $Y \in \mathscr{C}$ *and morphism* $f : X \to Y$*, we have:*

$$f = f \circ \mathrm{Id}_X = f \circ 0_X = 0_{XY}$$

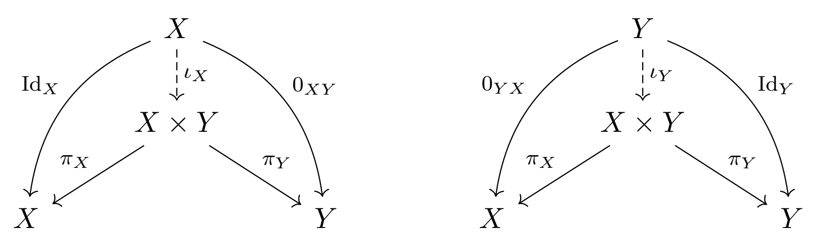
Hence, there is a unique morphism from $X \to Y$*. Similarly, there is a unique morphism from* $Y \to X$*. Hence,* $X \cong 0_{\mathscr{C}}$*.*

Example 11.7. ${}_R\mathsf{Mod}$ and Ab are additive categories.

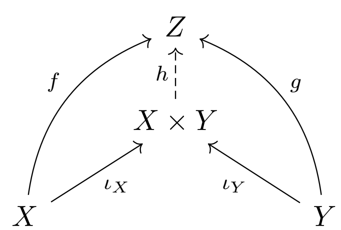
Some definitions of additive categories explicitly include the existence of finite products and co-products as axioms. However, it turns out that the existence of finite co-products can be deduced as a consequence.

Proposition 11.8. *Let* $\mathscr{C}$ *be an additive category. Finite co-products exist in* $\mathscr{C}$ *and they agree with products.*

Proof. Let $X, Y \in \mathscr{C}$, and consider their product, $X \times Y$. The universal property of the product gives morphisms ι_X, ι_Y such that the following diagram commutes:



We claim that $X \times Y$ together with ι_X, ι_Y form a co-product for X and Y. Given an object $Z \in \mathscr{C}$ and morphisms $f : X \to Z$ and $g : Y \to Z$, we need to show that there exists a unique morphism $h : X \times Y \to Z$ such that



commutes. To see such an h exists, consider $h := f \circ \pi_X + g \circ \pi_Y$. Then

$$\begin{aligned} h \circ \iota_X &= f \circ \pi_X \circ \iota_Y + g \circ \pi_Y \circ \iota_X \\ &= f \circ \mathrm{Id}_X + g \circ 0_{XY} \\ &= f, \end{aligned}$$

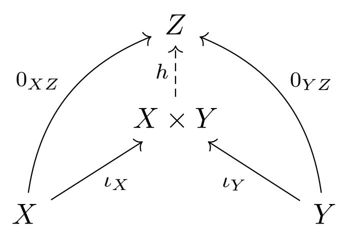
and

$$\begin{aligned} h \circ \iota_Y &= f \circ \pi_X \circ \iota_Y + g \circ \pi_Y \circ \iota_Y \\ &= f \circ 0_{YX} + g \circ \mathrm{Id}_Y \\ &= g. \end{aligned}$$

Suppose that h' is another morphism such that $h' \circ \iota_X = f$ and $h' \circ \iota_Y = g$. Then $h - h'$ satisfies

$$(h - h') \circ \iota_X = f - f = 0,$$
$$(h - h') \circ \iota_Y = g - g = 0.$$

So it is sufficient to show that the zero morphism is the unique morphism h such that the following diagram commutes:



First, we claim that $\iota_X \circ \pi_X + \iota_Y \pi_Y$ is the identity arrow on $X \times Y$. Indeed, this map satisfies

$$\pi_X \circ (\iota_X \circ \pi_X + \iota_Y \circ \pi_Y) = \pi_X \circ \iota_X + \pi_X \circ \iota_Y = \pi_X,$$
$$\pi_Y \circ (\iota_X \circ \pi_X + \iota_Y \circ \pi_Y) = \pi_Y \circ \iota_X + \pi_Y \circ \iota_Y = \pi_Y.$$

The identity morphism $\mathrm{Id}_{X \times Y}$ also satisfies the same conditions. So the universal property of the product guarantees that $\iota_X \circ \pi_X + \iota_Y \circ \pi_Y = \mathrm{Id}_{X \times Y}$. Now if $h \circ \iota_X = 0_{XZ}$ and $h \circ \iota_Y = 0_{YZ}$, then

$$\begin{aligned} h &= h \circ \mathrm{Id}_Z \\ &= h \circ (\iota_X \circ \pi_X + \iota_Y \circ \pi_Y) \\ &= h \circ \iota_X \circ \pi_X + h \circ \iota_Y \circ \pi_Y \\ &= 0_{XZ} \circ \pi_X + 0_{YZ} \circ \pi_Y = 0_{X \times Y, Z}. \end{aligned}$$

This completes the proof. □

Remark 11.9. *In what follows, for two objects $X, Y \in \mathscr{C}$, we use the notation $X \times Y$ and $X \oplus Y$ to interchangably represent its product and co-product.*

Proposition 11.10. *If $\mathscr{C}$ is an additive category, the object $X \oplus Y \cong X \times Y$ is characterized by the existence of morphisms*

$$X \underset{\pi_X}{\overset{i_X}{\rightleftarrows}} X \oplus Y \cong X \times Y \underset{i_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$$

such that

$$\pi_X \circ i_X = \mathrm{Id}_X,$$
$$\pi_Y \circ i_Y = \mathrm{Id}_Y,$$
$$\pi_Y \circ i_X = 0_X,$$
$$\pi_X \circ i_Y = 0_Y,$$
$$i_X \circ \pi_X + i_Y \circ \pi_Y = \mathrm{Id}_{X \oplus Y}.$$

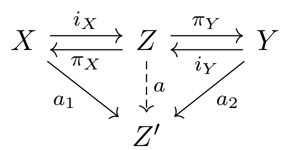
Proof. Let $Z \in \mathscr{C}$ such that along with the morphisms

$$X \underset{\pi_X}{\overset{i_X}{\rightleftarrows}} Z \underset{i_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$$

satisfying the conditions written in the statement of the theorem (with $X \oplus Y$ replaced by Z). We show that $Z \cong X \oplus Y$ by showing that Z satisfies the universal property for $X \oplus Y$. For $Z' \in \mathscr{C}$, let $a_X : X \to Z'$ and $a_Y : Y \to Z'$ be a pair of morphisms. Consider $a = a_X \circ \pi_X + a_Y \circ \pi_Y$ is a morphism from Z to Z'. Then:

$$a \circ i_X = a_X \circ \pi_X \circ i_X + a_Y \circ \pi_Y \circ i_X = a_X \circ \mathrm{Id}_X + a_Y \circ 0_Y = a_X.$$

Similarly, $a \circ i_Y = a_Y$.



Moreover, this morphism is unique. If a' is any other morphism, then

$$\begin{aligned} a' &= a' \circ \mathrm{Id}_Z \\ &= a' \circ (i_X \circ \pi_X + i_Y \circ \pi_Y) \\ &= a' \circ i_X \circ \pi_X + a' \circ i_Y \circ \pi_Y \\ &= a_1 \circ \pi_X + a_2 \circ \pi_Y = a. \end{aligned}$$

This completes the proof. □

12. Additive Functors

The correct notion of a functor between additive categories is that of an additive functor.

Definition 12.1. Let $\mathscr{C}$ and $\mathscr{D}$ be additive categories. An **additive functor** $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is a functor such that the map

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(X), \mathscr{F}(Y)), \\ f &\mapsto \mathscr{F}(f) \end{aligned}$$

is a homomorphism of abelian groups. That is for each $f, g \in \mathrm{Hom}(X, Y)$, we have

$$\mathscr{F}(f + g) = \mathscr{F}(f) + \mathscr{F}(g)$$

Example 12.2. The following are examples of additive functors:

(1) Let $\mathscr{C}$ be a locally small additive category and $A \in \mathscr{C}$. The $\mathrm{Hom}(A, -)$ functor may be viewed as functor into Ab. In fact, the $\mathrm{Hom}(A, -)$ functor is an example of an additve functor, Similar remarks apply to the contravariant Hom functor.

(2) Let R be a commutative ring and let N be a R-module. The tensor product, $- \otimes_R N$, defines an additive functor from ${}_R\mathsf{Mod}$ to ${}_R\mathsf{Mod}$.

Proposition 12.3. *Let $\mathscr{F} : \mathscr{C} \longrightarrow \mathscr{D}$ be an additive functor between additive categories. We have the following:*

(1) $\mathscr{F}(0_{\mathscr{C}}) = 0_{\mathscr{D}}$.

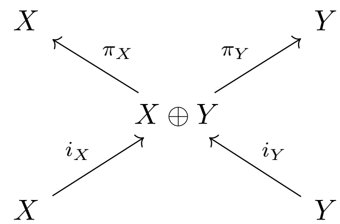
(2) *For any $X, Y \in \mathscr{C}$, we have $\mathscr{F}(X \xrightarrow{0_{XY}} Y) = \mathscr{F}(X) \xrightarrow{0_{\mathscr{F}(X)\mathscr{F}(Y)}} \mathscr{F}(Y)$.*

(3) *$\mathscr{F}$ preserves finite products and co-products.*

Proof. (1) Note that $\mathscr{F}(\mathrm{Id}_{0_{\mathscr{C}}}) = \mathrm{Id}_{\mathscr{F}(0_{\mathscr{C}})}$ and $\mathscr{F}(0_{0_{\mathscr{C}}}) = 0_{\mathscr{F}(0_{\mathscr{C}})}$ since it is a group homomorphism on the level of morphisms. Thus

$$\mathrm{Id}_{\mathscr{F}(0_{\mathscr{C}})} = \mathscr{F}(\mathrm{Id}_{0_{\mathscr{C}}}) = \mathscr{F}(0_{0_{\mathscr{C}}}) = 0_{\mathscr{F}(0_{\mathscr{C}})}$$

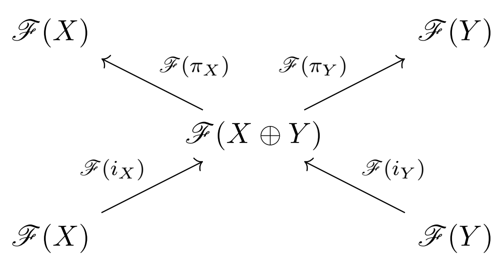
By Remark 11.6, we have $\mathscr{F}(0_{\mathscr{C}}) \cong 0_{\mathscr{D}}$. (2) is a straightforward consequence of (1). For (3), consider the following diagram:



such that

$$\begin{aligned} \pi_X \circ i_X &= \mathrm{Id}_X \\ \pi_Y \circ i_Y &= \mathrm{Id}_Y \\ \pi_Y \circ i_X &= 0_X \\ \pi_X \circ i_Y &= 0_Y \\ i_X \circ \pi_X + i_Y \circ \pi_Y &= \mathrm{Id}_{X \oplus Y} \end{aligned}$$

Consider the following diagram:



Clearly

$$\begin{aligned} \pi_{\mathscr{F}(X)} \circ i_{\mathscr{F}(X)} &= \mathrm{Id}_{\mathscr{F}(X)} \\ \pi_{\mathscr{F}}(Y) \circ i_{\mathscr{F}}(Y) &= \mathrm{Id}_{\mathscr{F}(Y)} \\ \pi_{\mathscr{F}(Y)} \circ i_{\mathscr{F}(X)} &= 0_{\mathscr{F}(X)} \\ \pi_{\mathscr{F}(X)} \circ i_{\mathscr{F}(Y)} &= 0_{\mathscr{F}(Y)} \\ i_{\mathscr{F}(X)} \circ \pi_{\mathscr{F}(X)} + i_{\mathscr{F}(Y)} \circ \pi_{\mathscr{F}(Y)} &= \mathrm{Id}_{\mathscr{F}(X \oplus Y)} \end{aligned}$$

By Proposition 11.10, we have that:

$$\mathscr{F}(X \oplus Y) \cong \mathscr{F}(X) \oplus \mathscr{F}(Y)$$

This completes the proof. □

Remark 12.4. *In what follows, when it is clear from context, we will write a zero morphism* $0_{XY} : X \to Y$ *as simply* 0.

13. Kernels & Cokernels

In an additive category, we can talk about monomorphisms and epimorphisms. Indeed, the definition of monomorphisms and epimorphisms is the same as those given in Section 2.1. However, exploiting the additional structure of an additive category, one can relax the definitions as follows.

Remark 13.1. *We use the phrase 'mono' for a monomorphism and the phrase 'epic' for an epimorphism.*

Lemma 13.2. *Let $\mathscr{C}$ be an additive category and consider morphisms $g : Z \to X, f : X \to Y$ and $h : Y \to Z'$ in $\mathscr{C}$. Then f is monic if and only if*

$$f \circ g = 0_{ZY} \Rightarrow g = 0_{ZX}$$

Similarly, f is epic if and only if

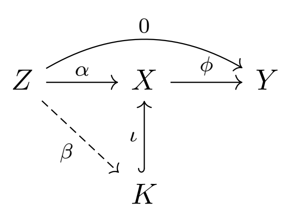
$$h \circ f = 0_{XZ'} \Rightarrow h = 0_{YZ'}$$

Proof. Each $\mathrm{Hom}(-,-)$ set is an abelian group. The conclusion follows because two morphisms with the same source and target are equal if and only if their difference in the corresponding Hom set is the zero morphism. ☐

One can also define kernels and cokernels in an additive category. Indeed, abstracting from our earlier discussion of kernels and cokernels of morphisms of R-modules, as presented in Part 2, we arrive at the following definition:

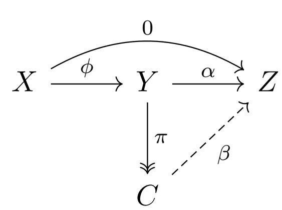
Definition 13.3. Let $\mathscr{C}$ be an additive category, and let $\phi : X \to Y$ be a morphism in $\mathscr{C}$.

(1) A morphism $\iota : K \to X$ is a **kernel** of ϕ if $\phi \circ \iota = 0$ and for all morphisms $\alpha : Z \to X$ such that $\phi \circ \alpha = 0$, there exists a unique $\beta : Z \to K$ making the diagram



commute.

(2) A morphism $\pi : Y \to C$ is a **cokernel** of ϕ if $\pi \circ \phi = 0$ and for all morphisms $\alpha : Y \to Z$ such that $\alpha \circ \phi = 0$, there exists a unique $\beta : C \to Z$ making the diagram



commute.

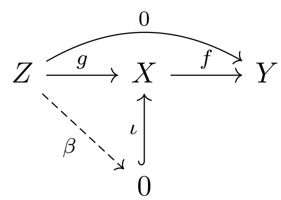
We have the following relationship between kernels and monomorphisms and cokernels and epimorphisms in an additive category.

Proposition 13.4. *Let $\mathscr{C}$ be an additive category, and let $\phi : X \to Y$ be a morphism in $\mathscr{C}$.*

(1) *If* $\ker f$ *exists, then* f *is monic if and only if ker* f *is the zero morphism* $0 \to X$*. In this case, we write* $\ker f = 0$
(2) *If* $\operatorname{coker} f$ *exists, then* f *is epic if and only if* $\operatorname{coker} u$ *is the zero morphism* $Y \to 0$*. In this case, we write* $\operatorname{coker} u = 0$*.*

Proof. The proof proceeds in the following steps:

(1) Assume that $\iota : 0 \to X$ is the kernel of f. If $g : Z \to X$ satisfies $f \circ g = 0$, then the universal property of the kernel provides a morphism $\beta : Z \to X$ with $g = \iota \circ \beta = 0$ (because $\iota = 0$). Hence, f is monic.



Conversely, assume f is monic. Denoting the kernel by the morphism $\iota : K \to X$, consider:

$$K \xrightarrow{\iota} X \xrightarrow{f} Y$$

Since $f \circ \iota = 0_{KY}$, we have that ι is the zero morphism onto X which must be $0 \to X$ by uniqueness.
(2) The proof is analogous to (1).

This completes the proof. □

14. Abelian Categories

Abelian categories naturally build upon the structure of additive categories, enriching them to provide a powerful framework for algebra and homological algebra. While additive categories allow for the addition of morphisms and the existence of finite biproducts, abelian categories go further by guaranteeing the existence of kernels and cokernels and supporting the formulation of exact sequences. This added structure makes abelian categories robust enough to encompass many familiar mathematical settings, such as categories of modules, sheaves, and chain complexes, thereby offering a unified approach to their study.

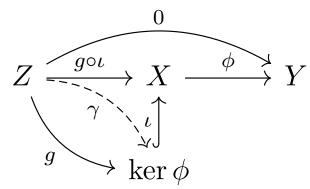
Definition 14.1. An additive category $\mathscr{C}$ is an **abelian category** if:

(1) Every morphism has a kernel and a cokernel,
(2) If ϕ is a monomorphism, then ϕ is the kernel of the $\operatorname{coker} \phi$.
(3) If ϕ is an epimorphism, then ϕ is the cokernel of $\ker \phi$.[15]

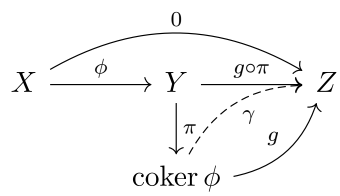
Proposition 14.2. *Let* $\mathscr{C}$ *be an abelian category*[16] *and let* $\phi : X \to Y$ *be a morphism. Then* $\ker \phi$ *is a monormophism and* $\operatorname{coker} \phi$ *is an epimorphism.*

[15]A monomorphism is normal if it is the kernel of some morphism, and an epimorphism is conormal if it is the cokernel of some morphism. A category $\mathscr{C}$ is binormal if it is both normal and conormal. In other words, an abelian category is binormal.
[16]This proposition is true in an additive category provided all kernels, cokernels and morphisms exist.

Proof. Let $\iota : \ker\phi \to X$ be the kernel of ϕ. Let $g : Z \to \ker\phi$ be a morphism such that $\iota \circ g = 0$. Then $\phi \circ (\iota \circ g) = 0$. Hence, there is a unique morphism $\gamma : Z \to \ker\phi$ such that the following diagram commutes:



Since $\iota \circ g = 0 = 0 \circ \iota$, the uniqueness of the morphism γ forces $g = 0$. Hence, $\ker\phi$ is a monomorphism. Similarly, let $\pi : Y \to \operatorname{coker}\phi$ be the cokernel of ϕ. Let $g : \operatorname{coker}\phi \to Z$ be a morphism such that $g \circ \pi = 0$. Then $\phi \circ (g \circ \pi) = 0$. Hence, there is a unique morphism $\gamma : \operatorname{coker}\phi \to Z$ such that the following diagram commutes:



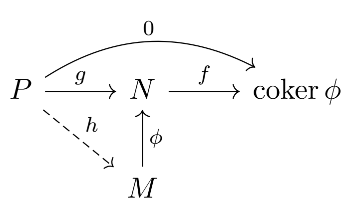
Since $g \circ \pi = 0 = 0 \circ \pi$, the uniqueness of the morhpism γ forces $g = 0$. Hence, $\operatorname{coker}\phi$ is an epimorphism. □

Remark 14.3. *In fact, Proposition 14.2 proves that in an abelian category, every kernel is the kernel of its cokernel. Similarly, every cokernel is the cokernel of its kernel. In an abelian category, we have the slogan:*

$$\text{'kernel} \iff \text{monic'}$$
$$\text{'cokernel} \iff \text{epic'}$$

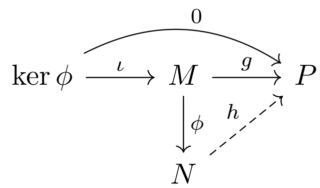
Proposition 14.4. ${}_R\mathsf{Mod}$ *is an abelian category.*

Proof. We already know that ${}_R\mathsf{Mod}$ is an additive category satisfying (1) in Definition 14.1. We verify (2) and (3). Let $\phi : M \to N$ be a morphism of modules such that ϕ is a monomorphism. Let $f : N \to \operatorname{coker}\phi$. We show that ϕ is the kernel of f. Note that $f \circ \phi = 0$. If $g : P \to N$ is another morphism such that $f \circ g = 0$, then $g(P) \subseteq \operatorname{im}\phi$. Since ϕ is a monomorphism, ϕ is injective. By the first isomorphism theorem, $M \cong \operatorname{im}\phi$. In particular, this implies there is a unique morphism $h : P \to M$ making the following diagram commute.



Let $\phi : M \to N$ be a morphism of modules such that ϕ is an epiomorphism. Let $\iota : \ker\phi \to M$ be the inclusion map. We show that ϕ is the cokernel of ι. Note that $\phi \circ \iota = 0$. If $g : M \to P$ is another morphism such that $g \circ \iota = 0$, then $\ker\phi \subseteq \ker g$. Since ϕ is an epimorphism, ϕ is surjective. By the first isomorphism theorem, $N \cong M/\ker\phi$. Since

$\ker\phi \subseteq \ker g$, this implies there is a unique morphism $h : N \to P$ making the following diagram commute.

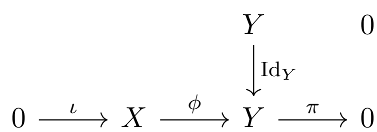


This completes the proof. □

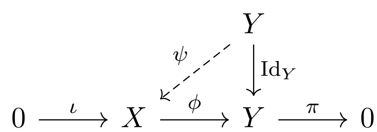
Why all the fuss about abelian categories? We argue that an abelian category is the correct generalization of ${}_R\mathsf{Mod}$. If a property holds in ${}_R\mathsf{Mod}$, we expect it to hold in any abelian category as well. For instance, we state the following sample proposition:

Proposition 14.5. *Let $\mathscr{C}$ be an abelian category, and let $\phi : X \to Y$ be a morphism in $\mathscr{C}$. If ϕ is both a monomorphism and an epimorphism, then ϕ is an isomorphism.*

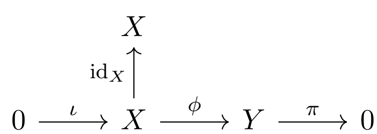
Proof. Since $\mathscr{C}$ is an abelian category, $\ker\phi$ and $\operatorname{coker}\phi$ exist. Since $\mathscr{C}$ is, in particular, an additive category and ϕ is both monic and epic, we have $\ker\phi = 0 = \operatorname{coker}\phi$. Hence $\ker\phi$ is $\iota : 0 \to X$ and $\operatorname{coker}\phi$ is $\pi : Y \to 0$. Further, ϕ is the cokernel of $0 \to X$ and the kernel of $Y \to 0$. Consider the following diagram:



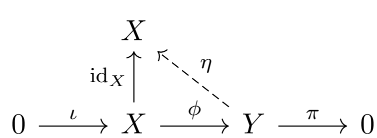
Trivially, $\pi \circ \mathrm{Id}_Y$ is the zero morphism. Since ϕ is the kernel of π, there is a unique morphism ψ making the diagram commute:



As $\phi \circ \psi = \mathrm{Id}_Y$, this shows that ϕ has a right-inverse. Similarly, consider the diagram:



Trivially, $\mathrm{Id}_X \circ \iota$ is the zero morphism. Since ϕ is the cokernel of ι, there is a unique morphism η making the diagram commute:



As $\eta \circ \phi = \mathrm{Id}_X$, this shows that ϕ has a left-inverse. Since ϕ has both a right-inverse and a left-inverse, $\psi = \eta$, implying that ϕ is an isomorphism. □

Further substantiating our claim about the similarly between ${}_R\mathsf{Mod}$ and an abelian category, we show that an analog of the first isomorphism theorem holds in an abelian category. To proves this claim, we first need to make the following definition:

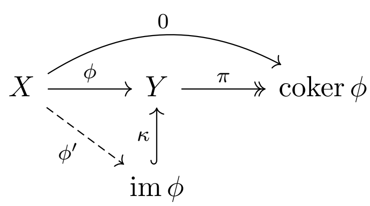
Definition 14.6. Let $\mathscr{C}$ be an abelian category. For $X, Y \in \mathscr{C}$, let $\phi : X \to Y$ be a morphism. Let $i : \ker\phi \to X$ and $\pi : Y \to \operatorname{coker}\phi$. Then

(1) The **image of** ϕ, denoted by im ϕ, is the kernel of π, denoted as the map $\kappa : \text{im } \phi \to Y$

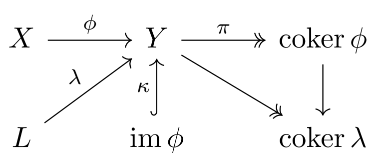
(2) The **coimage of** ϕ, denoted by coim ϕ, is the cokernel of ι, denote as the map $\rho : X \to \text{coim } \phi$.

Proposition 14.7. *Let $\phi : X \to Y$ be a morphism in an abelian category. Then ϕ factors though* $\operatorname{im}\phi$ *and* $\operatorname{im}\phi$ *is initial with this property. Similarly, Then ϕ factors out of* $\operatorname{coim}\phi$ *and* $\operatorname{coim}\phi$ *is final with this property.*

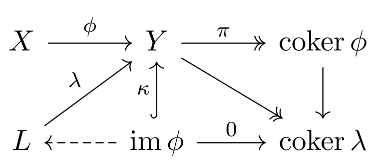
Proof. It is clear that ϕ factors through $\operatorname{im}\phi$: the composition $X \xrightarrow{\phi} Y \xrightarrow{\pi} \operatorname{coker}\phi$ is the zero-morphism, so there is a naturally induced $\phi' : X \to \operatorname{im}\phi$ by the universal property of kernels.



Now let $\lambda : L \to Y$ be any monomorphism through which ϕ factors. Since ϕ factors through λ, the composition $X \to Y \to \operatorname{coker}\lambda$ is 0. By universal property of $\operatorname{coker}\phi$, there is a unique map from $\operatorname{coker}\phi$ to $\operatorname{coker}\lambda$ making the following diagram commute:

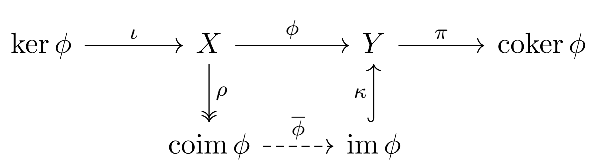


Since $\operatorname{im}\phi \to \operatorname{coker}\phi$ is the zero-morphism, this implies that $\operatorname{im}\phi \to \operatorname{coker}\lambda$ is the zero-morphism. Since λ is the kernel of $\operatorname{coker}\lambda$, there is a unique morphism $\operatorname{im}\phi \to L$ making the following diagram



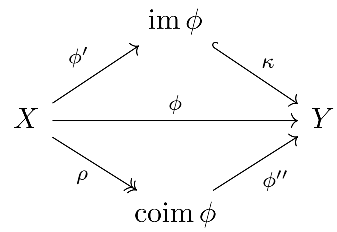
commute. The proof of this second claim is similar. □

Proposition 14.8. *Let $\mathscr{C}$ be an abelian category. For $X, Y \in \mathscr{C}$, let $\phi : X \to Y$ be a morphism, and let $\kappa : \operatorname{im}\phi \to Y$ and be the induced morphisms. There is a unique morphism $\overline{\phi} : \operatorname{coim}\phi \to \operatorname{im}\phi$ making the following diagram*



commute.

Proof. Proposition 14.7 implies that there exist unique morphisms $\phi' : X \to \operatorname{im}\phi$ and $\phi'' : \operatorname{coim}\phi \to Y$ making the following diagram



commute. In fact, it can be shown that ϕ' is an epimorphism and ϕ'' is a monomorphism[17]. A unique morphism $\psi : \operatorname{coim}\phi \to \operatorname{im}\phi$ by the universal property of $\operatorname{im}\phi$ and universal property of $\operatorname{coim}\phi$. Since $\phi'' \circ \psi = \operatorname{im}\phi$ is a monomorphism, so is ψ and since $\psi \circ \phi' = \operatorname{coim}\phi$ is an epimorphism, so is ψ. It follows that ψ is an isomorphism, and letting $\overline{\phi} : C \to K$ be the inverse of ψ concludes the proof. □

15. Homological Algebra

We now turn to exploring how classical homological algebra concepts can be developed and generalized within the framework of an abelian category, allowing for a broader and more abstract approach.

15.1. **Exact Sequences.** Let $\mathscr{C}$ be a small abelian category. Recall that the notions of a kernel, image and cokernel of a morphism can be defined in a small abelian category. This allows to formalize the meaning of exact sequence in a small abelian category.

Definition 15.1. Let $\mathscr{C}$ be a small abelian category. Consider a sequence of objects and morphisms in a small abelian category:

$$\cdots \to A_{n-1} \xrightarrow{\varphi_{n-1}} A_n \xrightarrow{\varphi_n} A_{n+1} \to \cdots$$

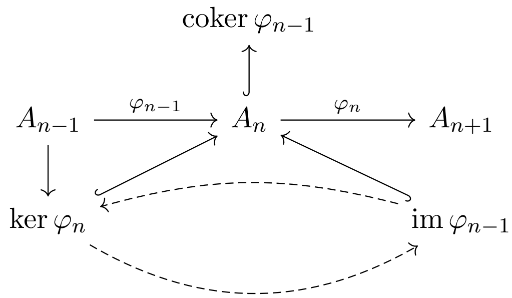
The sequence is **exact at** A_n if

(1) $\varphi_n \circ \varphi_{n-1} = 0$, and
(2) $\operatorname{coker}\varphi_{n-1} \circ \ker\varphi_n = 0$.

The sequence is **exact** if it is exact at each A_n for each $n \in \mathbb{Z}$. Such a sequence is called a long exact sequence.

What does exactness at A_n entail? The universal property of the kernel and the first condition tells us that φ_{n-1} factors through $\ker\varphi_n$. But φ_{n-1} also factors through $\operatorname{im}\varphi_{n-1}$. Using the universal property of images, there is a unique factorization of $\operatorname{im}\varphi_{n-1}$ through $\ker\varphi_n$. Similarly, the second condition tells us that $\ker\varphi_n$ uniquely factors through $\ker(\operatorname{coker}\varphi_{n-1}) = \operatorname{im}\varphi_{n-1}$. This implies that $\operatorname{im}\varphi_{n-1}$ and $\ker\varphi_n$ in the sense that these are isomorphic as subojects of A_n.

[17]This needs proof.



The conditions defining exactness can therefore be summarized as:

$$\operatorname{im}\varphi_{n-1} \cong \ker\varphi_{n-1}$$

Example 15.2. Let $\mathscr{C} = {}_R\mathsf{Mod}$. Then $\operatorname{im}\varphi$ and $\ker\psi$ are defined in the obvious way. In that case, a sequence

$$\cdots \to A_{n-1} \xrightarrow{\varphi_{n-1}} A_n \xrightarrow{\varphi_n} A_{n+1} \to \cdots$$

is exact at A_n if and only if $\operatorname{im}\varphi_{n-1} \cong \ker\varphi_{n-1}$ in the usual sense. The following is a list of examples of exact sequences in ${}_R\mathsf{Mod}$.

(1) Any left R-module M can be viewed as a sequence

$$\cdots \to 0 \to M \to 0 \to \cdots .$$

The maps are all the zero maps. The sequence is exact at M if and only if $M \cong 0$.

(2) Any R-module homomorphism $f : M \to N$ can be viewed as a sequence

$$\to \cdots \to 0 \to \ker f \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0 \to \cdots$$

Using the notion of exactness, we can rephrase familiar definitions from basic algebra.

Proposition 15.3. *Let $\mathscr{C}$ be a small abelian category.*

(1) *Consider the following exact sequence*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

(a) *ψ is a monomorphism if and only if φ is the zero-morphism.*
(b) *φ is an epiomorphism if and only if ψ is the zero-morphism.*

(2) *A sequence*

$$0 \xrightarrow{\varphi} A \xrightarrow{\psi} B$$

is exact if and only if ψ is a monomorphism.

(3) *A sequence*

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} 0$$

is exact if and only if ψ is an epimorphism.

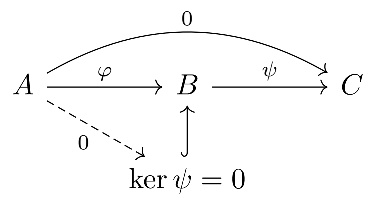
(4) *A sequence*

$$0 \to A \xrightarrow{\varphi} B \to\to$$

is exact if and only if φ is an isomorphism.

Proof. The proof is given below:

(1) It suffices to prove (a) since (b) is dual of (a). ψ is a monomorphism if and only if φ is the zero-morphism. If ψ is a monomorphism, then if $\ker\psi = 0$. By the universal property of kernel, there is a unique morphism $A \to 0$ making the diagram



commute. In fact, $A \to 0$ is the zero morphism. Moreover, $0 \to B$ is also the zero morphism. Hence, φ is the zero morphism. Conversely, if φ is the zero morphism then $\operatorname{im}\varphi \cong 0$. But the sequence is exact, so $\ker\psi \cong \operatorname{im}\varphi \cong 0$. Hence, ψ is a monomorphism.

(2) If the sequence is exact, then $\operatorname{im}\varphi \cong \ker\psi$. But $\operatorname{im}\varphi \cong 0$. Hence, $\ker\psi \cong 0$ and ψ is a monomorphism. Conversely, if ψ is a monomorphism, then $\ker\psi \cong 0$. Then $\operatorname{coker}\varphi \circ \ker\psi = 0$. Trivially, $\psi \circ \varphi = 0$. Hence, the sequence is exact.

(3) (3) is dual to (2) so it is clearly true.

(4) This follows by (3) and (4).

This completes the proof. □

Remark 15.4. *If $\mathscr{C} = {}_R\mathsf{Mod}$, we can give a direct proof of some statements in Proposition 15.3.*

(1) *f is injective if and only if $0 \to A \xrightarrow{f} B$ is exact. Indeed, the sequence is exact at A if and only if $\ker f = 0$ if and only if f is injective.*

(2) *f is surjective if and only if $A \xrightarrow{f} B \to 0$ is exact. Indeed, the sequence is exact at B if and only if $\operatorname{Im} f = B$ if and only if f is surjective.*

(3) *f is an isomorphism if and only if $0 \to A \xrightarrow{f} B \to 0$ is exact. This follows from the two statements above.*

Proposition 15.5. *Let $\mathscr{C}$ be a small abelian category. The following statements are true:*

(1) *A sequence*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

is exact if and only if φ is a monomorphism and φ is a kernel of ψ. Such a sequence is called a ***left short exact sequence****.*

(2) *A sequence*

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

is exact if and only if ψ is an epimorphism and ψ is a cokernel of φ. Such a sequence is called a ***right short exact sequence****.*

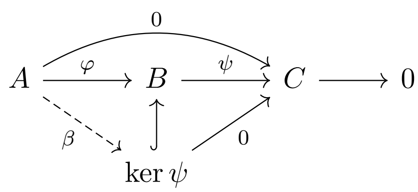
(3) *A sequence*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

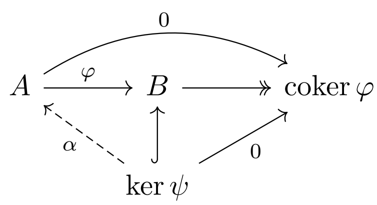
is exact if and only if φ is a kernel of ψ and ψ is a cokernel of φ. Such a sequence is called a ***short exact sequence****.*

Proof. The proof proceeds in the following steps:

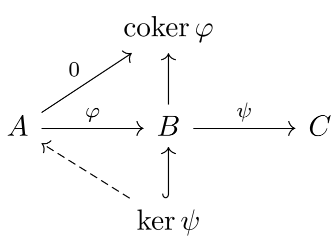
(1) Indeed, exactness at A is equivalent to φ being a monomorphism. Exactness at B is equivalent to $\psi \circ \varphi = 0$ and $\operatorname{coker} \varphi \circ \ker \psi = 0$. Assume the sequence is exact at B, the condition $\psi \circ \varphi = 0$ implies there is a unique morphism $\beta : A \to \ker \psi$ such that the diagram



commutes. Similarly, the conditions $\operatorname{coker} \varphi \circ \ker \psi = 0$ and φ is a monomorphism, which implies that φ is the kernel of $\operatorname{coker} \varphi$, implies that there is a unique morphism $\alpha : \ker \psi \to A$ such that the following diagram



commutes. Both α and β are easily shown to be inverses of each other. Hence, φ is a kernel of ψ. Conversely, assume that φ is a kernel of ψ. Then by the definition of kernel, $\varphi \circ \psi = 0$. Moreover by the universal property of the kernel, there exists a unique map $\ker \psi \to A$ making the diagram



commute. This readily implies that $\operatorname{coker} \varphi \circ \ker \psi = 0$.

(2) This statement is the dual of (1) so it is obviously true.

(3) The equivalence follows by (1) and (2).

This completes the proof. □

Remark 15.6. *If* $\mathscr{C} = {}_R\mathsf{Mod}$*, then a sequence*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

is exact if and only if φ *is injective,* ψ *is surjective and* $\operatorname{im} \varphi = \ker \psi$*. One can give a direct proof without invoking categorical arguments.*

Example 15.7. Let $\mathscr{C} = {}_{\mathbb{Z}}\mathsf{Mod} = \mathsf{Ab}$. Consider the following sequence of $\mathbb{Z}$-modules:

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

The first non-trivial map is multiplication by n, and the second non-trivial map is the quotient map. Multiplication by n is injective, and the image of such a map is the abelian group $(n) \subseteq \mathbb{Z}$. The kernel of the projection morphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the abelian group $(n) \subseteq \mathbb{Z}$. Hence, the sequence is a short exact sequence.

We now define the notion of a split short exact sequence:

Definition 15.8. Let $\mathscr{C}$ be a small abelian category. A short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a **split short exact sequence** if there exists an isomorphism $B \to A \oplus C$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle w} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{\ \iota_A\ } & A \oplus C & \xrightarrow{\ \pi_C\ } & C & \longrightarrow & 0 \end{array}$$

commutes, where $\iota_A : A \to A \oplus C$ and $\iota_C : C \to A \oplus C$ are the inclusion and projection morphisms respectively generated by thinking of $A \oplus C$ as a co-product and product respectively.

Example 15.9. Let $R = \mathbb{Z}$ and $\mathscr{C} = {}_{\mathbb{Z}}\mathsf{Mod} = \mathsf{Ab}$ be the category of abelian groups. The short exact sequence

$$0 \xrightarrow{\iota} \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$$

where $\iota(x) = (x, 0)$ and $\pi(x, y) = y$ is a split short exact sequence. The short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$$

where π is the canonical project is not split exact since $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Remark 15.10. *In what follows, we will often abbreviate a short exact sequence as SES.*

We have seen that not all short exact sequences are split short exact sequences in a small abelian category. When does a SES split, though? We have the following criterion:

Proposition 15.11. *Let $\mathscr{C}$ be a small abelian category. Consider the SES:*

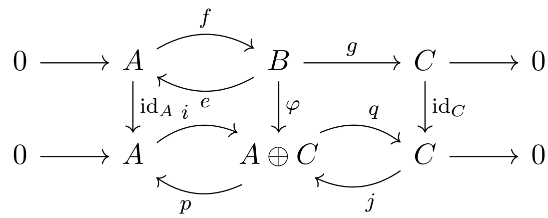
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

The following are equivalent:

1. *The SES splits.*
2. *f is a split monomorphism. That is, there is a morphism $e : B \to A$ such that $e \circ f = \mathrm{id}_A$,*
3. *g is a split epimorphism. That is, there is a morphism $h : C \to B$ such that $g \circ h = \mathrm{id}_C$.*

Proof. Clearly, (1) implies (2) and (3). We show that (2) implies (1). Let $i : A \to A \oplus C$, $j : C \to A \oplus C$, $p : A \oplus C \to A$, $q : A \oplus C \to C$ be the 4 morphisms characterizing $A \oplus C$.

Consider the following diagram:



ϕ is defined as the unique morphism such that $e = p \circ \varphi$, $g = q \circ \varphi$. It's easy to show that this diagram commutes. By the short-five lemma Lemma 15.13, ϕ is an isomorphism. A similar argument shows that (3) implies (1). $\square$

Remark 15.12. *Proposition 15.11 is not necessarily true in a non-abelian category. Let $\mathscr{C} = \mathsf{Grp}$, the category of groups. Consider the following sequence in Grp:*

$$0 \to A_3 \xrightarrow{i} S_3 \xrightarrow{\mathrm{sgn}} \mathbb{Z}/2\mathbb{Z} \to 0.$$

We can define a group homomorphism $h : \mathbb{Z}/2\mathbb{Z} \to S_3$ by mapping 1 to $(1, 2)$. Clearly, $\mathrm{sgn} \circ h = \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}}$. However, the sequence is not SES. This is because S_3 is a non-abelian group and $A_3 \oplus \mathbb{Z}/2\mathbb{Z}$ is an abelian group.

With the notion of an exact sequence in an abelian category, we are now in a position to discuss homological algebra proper in a small abelian category. In particular, one can start pasting together various exact sequences to prove a number of diagram chasing results. We now prove the short five lemma, which we have already invoked above.

Lemma 15.13. ***(Short Five Lemma)*** *Let $\mathscr{C}$ be a small abelian category. Consider the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

The following statements are true:

1. *If α and γ are monorphisms, then β is a monormorphism;*
2. *If α and γ are epimorphisms, then β is an epimorphism;*
3. *If α and γ are isomorphisms, then β is an isomorphism.*

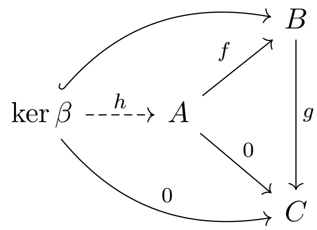
Proof. The proof proceeds as follows:

(1) Consider the following augmented diagram:

$$\begin{array}{ccccccccc} & & 0 = \ker\alpha & \longrightarrow & \ker\beta & \longrightarrow & \ker\gamma = 0 & & \\ & & \downarrow{\scriptstyle\iota_\alpha} & & \downarrow{\scriptstyle\iota_\beta} & & \downarrow{\scriptstyle\iota_\gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

The horizontal arrows in the top row are the unique zero morphism. Note that $g \circ \iota_\beta = 0$ by the commutativity of the upper right square. Since f is the kernel of

g by Proposition 15.5 (3), we have a commutative diagram:



Therefore,

$$f' \circ \alpha \circ h = \beta \circ f \circ h = \beta \circ \iota_\beta = 0$$

Since f' and α are monomorphisms, $f' \circ \alpha$ is a monomorphism. Hence, $h = 0$. This shows that $\ker \beta = f \circ h = 0$, which implies that β is a monomorphism.

(2) This statement is the dual of (1). Hence, it is true.
(3) This statement from (2) and (3).

This completes the proof. □

15.2. **(Co)-Chain Complexes.** It is easy to come up with examples of sequences that are not exact.

Example 15.14. Let $\mathscr{C} = {}_{\mathbb{Z}}\mathsf{Mod} = \mathsf{Ab}$. Consider the following sequence of $\mathbb{Z}$-modules:

$$0 \to \mathbb{Z} \xrightarrow{2n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

The first non-trivial map is now multiplication by $2n$, that has image $(2n) \subseteq \mathbb{Z}$. The kernel of the projection morphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the abelian group $(n) \subseteq \mathbb{Z}$. We have $(2n) \subsetneq (n)$. Hence, the sequence is not a short exact sequence.

This motivates the following definition:

Definition 15.15. Let $\mathscr{C}$ be a small abelian category. A **chain complex** $(C_\bullet, d_\bullet)$ is a sequence of objects and morphisms,

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

such that, $d_n \circ d_{n+1} = 0$ for each $n \in \mathbb{Z}$. Similarly, a **co-chain complex** $(C_\bullet, d_\bullet)$ is a sequence of objects and morphisms,

$$\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \cdots$$

such that, $d^n \circ d^{n-1} = 0$ for each $n \in \mathbb{Z}$.

Example 15.16. Let $\mathscr{C} = \mathsf{Mod}_{\mathbb{Z}} = \mathsf{Ab}$. Consider the following sequence:

$$0 \xrightarrow{d^{-1}} \mathbb{Z} \xrightarrow{d^0} \mathbb{Z} \xrightarrow{d^1} 0$$

Here d^1 and d^{-1} are the zero maps at d^0 is the multiplication by two map. Clearly,

$$d^0 \circ d^{-1} = 0 \quad d^1 \circ d^0 = 0$$

We have $\ker d^0 = \{0\}$ and $\operatorname{Im} d^{-1} = \{0\}$. Hence, the sequence is exact at C^0. Moreover, $\ker d^1 = \mathbb{Z}$ and $\operatorname{Im} d^0 = 2\mathbb{Z}$. Hence, the sequence is not exact at at C^1.

Given a small abelian category, we have defined the notion of chain and co-chain complexes. Do chain and co-chain complexes form a small abelian category themselves? The answer is yes: the objects in the category will be chain or co-chain complexes and morphisms will be morphisms between chain and co-chain complexes as defined below:

Definition 15.17. Let $\mathscr{C}$ be a small abelian category. The **category of chain complexes**, denoted as $\mathsf{Chain}^{\mathscr{C}}$, contains objects as chain complexes and a morphism, $\alpha_\bullet$, between chain complexes $(C_\bullet, d_\bullet)$ and $(D_\bullet, e_\bullet)$ is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n+1}} & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{e_{n+1}} & D_n & \xrightarrow{e_n} & D_{n-1} & \longrightarrow & \cdots \end{array}$$

in $\mathscr{C}$. Similarly, the **category of co-chain complexes**, denoted as $\mathsf{CoChain}^{\mathscr{C}}$, contains objects as co-chain complexes and a morphism, $\alpha^\bullet$, between co-chain complexes $(C^\bullet, d^\bullet)$ and $(D^\bullet, e^\bullet)$ is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{n-1} & \xrightarrow{d^{n-1}} & C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha^{n-1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha^{n+1}} & & \\ \cdots & \longrightarrow & D^{n-1} & \xrightarrow{e^{n-1}} & D^n & \xrightarrow{e^n} & D^{n+1} & \longrightarrow & \cdots \end{array}$$

in $\mathscr{C}$.

Remark 15.18. *It is a simple matter to check that the above definition indeed defines relevant categories.*

Example 15.19. A commutative diagram in $\mathsf{Chain}^{\mathscr{C}}$ is called a chain map. Let $\mathscr{C} = \mathsf{Mod}_{\mathbb{Z}} = \mathsf{Ab}$. Consider the diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{4} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Each horizontal row we think of representing a complex. As the diagrams commute we have that the diagram represents a chain map of chain complexes. Similarly, a commutative diagram in $\mathsf{CoChain}^{\mathscr{C}}$ is called a co-chain map.

Proposition 15.20. *Let $\mathscr{C}$ be a small abelian category. Then $\mathsf{Chain}^{\mathscr{C}}$ is a small abelian category. Similarly, $\mathsf{CoChain}^{\mathscr{C}}$ is a small abelian category.*

Proof. (Sketch) We check that $\mathsf{CoChain}^{\mathscr{C}}$ is a small abelian category. The proof proceeds as follows:

(1) We can add morphisms of co-chain complex: if $\alpha^\bullet$ and $\beta^\bullet$ are morphisms between $(C^\bullet, d^\bullet)$ and $(D_\bullet, e_\bullet)$, their sum, $\alpha^\bullet + \beta^\bullet$ is defined by the family of maps $\{\alpha^n + \beta^n\}_{n\in\mathbb{Z}}$.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n-1} & \xrightarrow{d^{n-1}} & C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & \cdots \\ & & {\scriptstyle \alpha^{n-1}+\beta^{n-1}}\downarrow & & \downarrow{\scriptstyle \alpha^n+\beta^n} & & \downarrow{\scriptstyle \alpha^{n+1}+\beta^{n+1}} & & \\ \cdots & \longrightarrow & D^{n-1} & \xrightarrow{e^{n-1}} & D^n & \xrightarrow{e^n} & D^{n+1} & \longrightarrow & \cdots \end{array}$$

Noting that

$$\begin{aligned} e^n \circ (\alpha^n + \beta^n) &= e^n \circ \alpha^n + e^n \circ \beta_n \\ &= \alpha^{n+1} \circ d^n + \beta^{n+1} \circ d^n \\ &= (\alpha^{n+1} + \beta^{n+1}) \circ d^n \end{aligned}$$

for $n \in \mathbb{Z}$, it is clear that $\alpha^\bullet + \beta^\bullet$ is co-chain map.

(2) The zero object in $\mathsf{CoChain}^{\mathscr{C}}$ is the complex $(0^{\mathscr{C}}, 0^\bullet)$ where each object is the zero object, $0_{\mathscr{C}}$, in $\mathscr{C}$ and the maps are the zero morphisms:

$$\cdots \xrightarrow{0} 0_{\mathscr{C}} \xrightarrow{0} 0_{\mathscr{C}} \xrightarrow{0} \cdots$$

This is indeed the zero object as is evident from the following diagram:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{0} & 0_{\mathscr{C}} & \xrightarrow{0} & 0_{\mathscr{C}} & \xrightarrow{0} & 0_{\mathscr{C}} & \xrightarrow{0} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & C_{n-1} & \xrightarrow{d^{n-1}} & C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{0} & 0_{\mathscr{C}} & \xrightarrow{0} & 0_{\mathscr{C}} & \xrightarrow{0} & 0_{\mathscr{C}} & \xrightarrow{0} & \cdots \end{array}$$

(3) Given a family $\{(C_j^\bullet, d_j^\bullet)\}_{j=1}^n$ of co-chain complexes, the co-product of the co-chain complexes, denoted

$$\bigoplus_{j=1}^n (C_j^\bullet, d_j^\bullet)$$

is defined component-wise:

$$\cdots \longrightarrow \bigoplus_{j=1}^n C_j^{n-1} \xrightarrow{\oplus_{j=1}^n d_j^{n-1}} \bigoplus_{j=1}^n C_j^n \xrightarrow{\oplus_{j=1}^n d_j^n} \bigoplus_{j=1}^n C_j^{n+1} \longrightarrow \cdots$$

where

$$\bigoplus_{j=1}^n d_j^n = (d_1^n, \cdots, d_j^n)$$

for each $n \in \mathbb{Z}$. It is easy to check that this is a co-product in the category of co-chain complexes, since the universal property of the co-product is satisfied component-wise.

(4) If $\alpha^\bullet$ is a morphism between $(C^\bullet, d^\bullet)$ and $(D^\bullet, e^\bullet)$, then the co-chain complexes $(\ker C^\bullet, c^\bullet)$ and $(\text{coker} D^\bullet, f^\bullet)$ are defined by the diagram below:

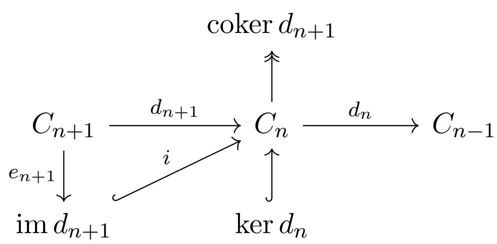
$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \ker C^{n-1} & \xrightarrow{c^{n-1}} & \ker C^n & \xrightarrow{c^n} & \ker C^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \ker \alpha^{n-1}} & & \downarrow{\scriptstyle \ker \alpha^n} & & \downarrow{\scriptstyle \ker \alpha^{n+1}} & & \\ \cdots & \longrightarrow & C^{n-1} & \xrightarrow{d^{n-1}} & C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha^{n-1}} & & \downarrow{\scriptstyle \alpha^n} & & \downarrow{\scriptstyle \alpha^{n+1}} & & \\ \cdots & \longrightarrow & D^{n-1} & \xrightarrow{e^{n-1}} & D^n & \xrightarrow{e^n} & D^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \text{coker}\alpha^{n-1}} & & \downarrow{\scriptstyle \text{coker}\alpha^n} & & \downarrow{\scriptstyle \text{coker}\alpha^{n+1}} & & \\ \cdots & \longrightarrow & \text{coker} D^{n-1} & \xrightarrow{f^{n-1}} & \text{coker} D^n & \xrightarrow{f^n} & \text{coker} D^{n+1} & \longrightarrow & \cdots \end{array}$$

The existence of the morphisms c^n's and f^n's is guaranteed by the universal property of kernels and cokernels. The universal properties of the kernel and cokerel co-chain complexes are easily seen to be true since they are true component-wise.

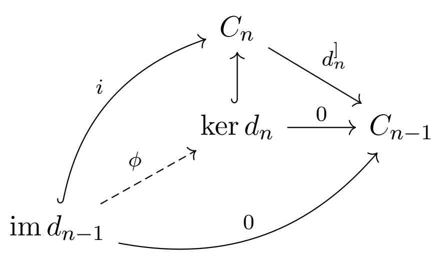
(5) If $\alpha^\bullet$ is a monormorphim (an epimorphism), then α^n is a monomorphism (an epimorphism) for each $n \in \mathbb{Z}$. Since each α^n is the kernel of the cokernel (cokernel of the kernel) of α^n, we have that $\alpha^\bullet$ is the kernel (cokernel) of coker$\alpha^\bullet$ ($\ker \alpha^\bullet$).

A similar argument shows that $\mathsf{Chain}^{\mathscr{C}}$ is a small abelian category. This completes the (sketch of the) proof. □

15.3. **(Co)homology.** Given a chain complex, $(C_\bullet, d_\bullet)$, the condition $d_n \circ d_{n+1} = 0$ implies that there is a unique morphism $\operatorname{im} d_{n+1} \to \ker d_n$.



Since $(C_\bullet, d_\bullet)$ is a chain complex, $d_n \circ d_{n+1} = 0$. Since $i \circ e_{n+1} = d_{n+1}$, we have that $d_n \circ i \circ e_{n+1} = 0$. Since e_{n+1} is an epimorphism, this implies that $d_n \circ i = 0$. Then by the universal property of kernels, there is a unique map $\varphi : \operatorname{im} d_{n-1} \to \ker d_n$ such that the following diagram



commutes. Similarly, if $(C^\bullet, d^\bullet)$ is a co-chain complex, the condition $d^n \circ d^{n-1} = 0$ implies that unique morphism $\operatorname{im} d^{n-1} \to \ker d^n$. This motivates the following definition:

Definition 15.21. Let $\mathscr{C}$ be a small abelian category.

(1) Let $(C_\bullet, d_\bullet)$ be a co-chain complex in $\mathscr{C}$. The **n-th homology** is the cokernel of the morphism $\operatorname{im} d_{n+1} \to \ker d_n$. It is denoted as:

$$H_n(C_\bullet) := \frac{\ker d_n}{\operatorname{im}\ d_{n+1}}.$$

(2) Let $(C^\bullet, d^\bullet)$ be a co-chain complex in $\mathscr{C}$. The **n-th cohomology** is the cokernel of the morphism $\operatorname{im} d_{n-1} \to \ker d_n$. It is denoted as:

$$H_n(C_\bullet) := \frac{\ker d^n}{\operatorname{im}\ d^{n-1}}.$$

Remark 15.22. *In what follows, we will work with either homology or cohomology. Working with the other yields similar definitions and results, which we will not repeat if an appropriate result has been stated for cohomology.*

Remark 15.23. *A chain complex* $(C_\bullet, d_\bullet)$ *such that* $H_n(C_\bullet)$ *is the zero object for each* $n \in \mathbb{Z}$ *that has homology is an exact sequence. Similar remarks apply to the cohomology case.*

Remark 15.24. *Let* R *be a ring* $\mathscr{C} = {}_R\mathsf{Mod}$ *and let* $(C_\bullet, d_\bullet)$ *be a co-chain complex. In this case, the definition of he* n*-th homology should be interpreted as follows: both* $\ker d^n$ *and* $\operatorname{im} d^{n+1}$ *are* R*-submodules of the module* C_n *Since* $\operatorname{im} d^{n+1} \subseteq \ker d^n \subset C^n$*, the quotient module*

$$H_n(C_\bullet) = \frac{\ker d^n}{\operatorname{im} d^{n+1}}$$

is well-defined and is the n*-th cohomology of* $(C_\bullet, d_\bullet)$*. Elements of* $\ker d^n$ *are called* n*-cycles and elements of* $\operatorname{im} d^{n+1}$ *are called* n*-boundaries. Similar remarks apply to the* n*-th cohomology of a co-chain complex.*

Example 15.25. Let $\mathscr{C} = \mathsf{Mod}_{\mathbb{Z}} = \mathsf{Ab}$. Consider the chain complex

$$C_\bullet : \cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} 0 \to \cdots,$$

where the chain groups are given by

$$C_1 = \mathbb{Z}, \quad C_2 = \mathbb{Z} \oplus \mathbb{Z}, \qquad C_n = 0 \text{ for } n \neq 1, 2,$$

The homomorphism d_2 is defined by $d_2(x, y) = 3x + 3y$. Note that we have the following

$$\operatorname{Ker} d_n \cong \begin{cases} \mathbb{Z}, & \text{if } n = 1 \text{ or } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

Similarly, we have

$$\operatorname{Im} d_n \cong \begin{cases} 3\mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \neq 2. \end{cases}$$

Hence

$$H_n(C_\bullet) \cong \begin{cases} \mathbb{Z}_3, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

Proposition 15.26. *Let* $\mathscr{C}$ *be a small abelian category. For every* $n \in \mathbb{Z}$*, the assignment*

$$H_n : (C_\bullet, d_\bullet) \mapsto H_n(C_\bullet)$$

defines a covariant functor $\mathsf{Chain}^{\mathscr{C}} \to \mathscr{C}$*.*

Proof. Skipped. □

Remark 15.27. *Similarly, for every* $n \in \mathbb{Z}$*, the assignment*

$$H^n : (C^\bullet, d^\bullet) \mapsto H^n(C^\bullet)$$

defines a covariant functor $\mathsf{CoChain}^{\mathscr{C}} \to \mathscr{C}$*.*

Part 4. **References**

References

[Rie17] Emily Riehl. *Category theory in context*. Courier Dover Publications, 2017 (cit. on pp. 1, 34).

[Lei14] Tom Leinster. *Basic category theory*. Vol. 143. Cambridge University Press, 2014 (cit. on p. 1).

[Mac13] Saunders Mac Lane. *Categories for the working mathematician*. Vol. 5. Springer Science & Business Media, 2013 (cit. on p. 1).

[Alu21] Paolo Aluffi. *Algebra: chapter 0*. Vol. 104. American Mathematical Soc., 2021 (cit. on p. 1).