COMPLEX GEOMETRY

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ABSTRACT. These are notes on some topics in complex geometry. I wrote these notes in graduate school while self-learning the subject. There may be typos. Please send corrections to junaid.aftab1994@gmail.com.

CONTENTS

Part 1. Preliminaries

1. SEVERAL COMPLEX VARIABLES

A complex manifold is modeled as a topological space where each open subset resembles an open subset of $\mathbb{C}^n$. Therefore, we first study some fundamentals of several complex variables theory.

1.1. **Definitions.** A complex differentiable function $f : \mathbb{C}^n \to \mathbb{C}$ is called analytic or holomorphic. Recall that if $n = 1$, a holomorphic function in one variable admits a local representation in terms of convergent power series. The purpose of this section is to discuss the case $n > 1$ and to elucidate both the similarities and differences across the cases.

Remark 1.1. *Let $p \in \mathbb{C}^n$. We will find it convenient to consider open discs with respect to the supremum norm in $\mathbb{C}^n$:*

$$\Delta_\varepsilon(p) := \{z \in \mathbb{C}^n : |z^k - p^k| < \varepsilon \text{ for } k = 1, 2, \dots, n\}$$

Here $|\cdot|$ is the usual metric on $\mathbb{C}$. Such a $\Delta_\varepsilon(p)$ is called a polydisc of radius ε around p. Let $\langle,\rangle_{\mathbb{C}^n}$ denote the inner product on $\mathbb{C}^n$ and let $\|\cdot\|_{\mathbb{C}^n}$ denote the corresponding norm. We denote the open ball of radius ε around $p \in \mathbb{C}^n$ as

$$\mathbb{B}_\varepsilon(p) := \{z \in \mathbb{C}^n \mid \|z - a\|_{\mathbb{C}^n} < \varepsilon\}$$

If $n = 1$, we denote $\mathbb{B}_\varepsilon(p)$ as $\mathbb{D}_\varepsilon(p)$. Note that we have,

$$\Delta_\varepsilon(p) = \mathbb{D}_\varepsilon(p^1) \times \cdots \times \mathbb{D}_\varepsilon(p^n)$$

The notion of a holomorphic function of one variable can be extended in a straightforward way.

Definition 1.2. Let $U \subseteq \mathbb{C}^n$ be an open subset. A function $f : U \to \mathbb{C}$ is a **holomorphic** in U if for each $p = (p^1, \dots, p^n) \in U$, it it continuous at p and the partial derivatives

$$\frac{\partial f}{\partial z^j}(p) = \lim_{\xi \to 0} \frac{f(p^1, \dots, p^j + \xi + \cdots, p^n) - f(p^1, \dots, p^n)}{\xi}, \qquad \xi \in \mathbb{C} \setminus \{0\}$$

exist for each $j \in \{1, \dots, n\}$. The limit over some punctured polydisc centered at the origin in $\mathbb{C}$.

Remark 1.3. *More generally, a vector-valued function $f : U \to \mathbb{C}^k$ is said to be holomorphic if each of its component functions is holomorphic.*

Remark 1.4. *If $n = 1$, it can be easily shown that the continuity assumption can be removed from Definition 1.2. In fact, the continuity assumption can be removed if $n > 1$: Hartog (1906) proved that a function that has complex partial derivatives at every point of an open subset of $\mathbb{C}^n$ is automatically continuous. The proof is involved.*

In one complex variable, there are several equivalent ways to characterize holomorphic functions. There are similar equivalent characterizations for holomorphic functions of several variables. We first generalize Cauchy's integral formula to several variables:

Lemma 1.5. *Let $\Delta_\varepsilon(p)$ be polydisc in $\mathbb{C}^n$. Let $f : \overline{\Delta_\varepsilon(p)} \to \mathbb{C}$ be a continuous function such that f is holomorphic with respect to every single component z^i at any point of $\Delta_\varepsilon(p)$. Then for any $z \in \Delta_\varepsilon(p)$, we have:*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta^1 - p^1| = \varepsilon} \cdots \int_{|\zeta^n - p^n| = \varepsilon} \frac{f(\zeta^1, \dots, \zeta^n)}{(\zeta^1 - z^1) \cdots (\zeta^n - z^n)} \, d\zeta^1 \cdots d\zeta^n.$$

Proof. Repeated application of the Cauchy integral formula in one variable yields

$$\begin{aligned}
f(z^1, \dots, z^n) &= \frac{1}{2\pi i} \int_{|\zeta^n - p^n| = \varepsilon} \frac{f(z^1, \dots, z^{n-1}, \zeta^n)}{\zeta^n - z^n} \, d\zeta^n \\
&= \frac{1}{(2\pi i)^2} \int_{|\zeta^n - p^n| = \varepsilon} \int_{|\zeta^{n-1} - p^{n-1}| = \varepsilon} \frac{f(z^1, \dots, \zeta^{n-1}, \zeta^n)}{(\zeta^n - z^n)(\zeta^{n-1} - z^{n-1})} \, d\zeta^{n-1} d\zeta^n \\
&\vdots \\
&= \frac{1}{(2\pi i)^n} \int_{|\zeta^n - p^n| = \varepsilon} \cdots \int_{|\zeta^1 - p^1| = \varepsilon} \frac{f(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n) \cdots (\zeta^1 - z^1)} \, d\zeta^1 \cdots d\zeta^n
\end{aligned}$$

This completes the proof. □

Remark 1.6. *If we let* $\partial\Delta_\varepsilon(p^i) = \{\zeta \in \mathbb{C} \mid |\zeta^i - p^i| = \varepsilon\}$ *and*

$$\Gamma_\varepsilon(p) := \partial\Delta_\varepsilon(p^1) \times \cdots \times \partial\Delta_\varepsilon(p^n)$$

Fubini's theorem implies that the integral in Lemma 1.5 can be written as

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma_\varepsilon(p)} \frac{f(\zeta^1, \ldots, \zeta^n)}{(\zeta^1 - z^1) \cdots (\zeta^n - z^n)} \, d\zeta^1 \cdots d\zeta^n.$$

Lemma 1.5 implies an important point about holomorphic functions in several variables. The value of f *on* $\Delta_\varepsilon(p)$ *is completely determined by the values of* f *on the set* $\Gamma_\varepsilon(p)$*, which is much smaller than the boundary of the polydisc* $\partial\Delta_\varepsilon(p)$*!*

Proposition 1.7. *(**Osgood's Lemma**) Let* $U \subseteq \mathbb{C}^n$ *be an open set and* $f : U \to \mathbb{C}$ *is a continuous function. The following are equivalent:*

(1) f *is holomorphic.*
(2) *If* $f(z) = u(z) + iv(z)$*, then* f *is smooth and satisfies the following Cauchy–Riemann equations:*

$$\frac{\partial u}{\partial x^j} = \frac{\partial v}{\partial y^j}, \quad \frac{\partial u}{\partial y^j} = -\frac{\partial v}{\partial x^j}, \quad j = 1, \ldots, n,$$

where $z^j = x^j + iy^j$*.*
(3) *For each* $p = (p^1, \ldots, p^n) \in U$*, there exists a neighborhood of* p *in* U *on which* f *is equal to the sum of an absolutely convergent power series of the form*

$$f(z) = \sum_{k_1, \ldots, k_n = 0}^{\infty} a_{k_1 \ldots k_n} (z^1 - p^1)^{k_1} \cdots (z^n - p^n)^{k_n}.$$

Proof. The proof is given below:

- (1) $\iff$ (2): Assume (1) is true. Because f is holomorphic in each variable separately, complex variable theory shows that it satisfies the Cauchy–Riemann equations with respect to each variable. If $p \in U$, let $\varepsilon > 0$ such that $\overline{\Delta_\varepsilon(p)} \subseteq U$. Smoothness now follows from Lemma 1.5. This is because we can repeatedly differentiate under the integral sign because $\overline{\Delta_\varepsilon(p)}$ is compact and the integrand is smooth. Hence, (1) is also true. Conversely, if (2) is true, then it is certainly continuous, and complex variable theory implies that it has a complex derivative with respect to each variable. Hence, (1) is also true.
- (1) $\iff$ (3): Assume (1) (and hence (2)) is true. Note that

$$\frac{1}{\zeta^j - z^j} = \frac{1}{(\zeta^j - p^j) - (z^j - p^j)} = \frac{1}{\zeta^j - p^j} \cdot \frac{1}{1 - \frac{z^j - p^j}{\zeta^j - p^j}},$$

Since $\frac{|z^j - p^j|}{|\zeta^j - p^j|} < 1$ on the domain of integration in the integral in Lemma 1.5, we can expand the last fraction on the right in a power series to obtain

$$\frac{1}{\zeta^j - z^j} = \frac{1}{\zeta^j - p^j} \sum_{k=0}^{\infty} \left(\frac{z^j - p^j}{\zeta^j - p^j} \right)^k,$$

This power series converges uniformly and absolutely for z^j in any closed polydisk $\Delta_{\varepsilon'}(p^j)$ with $0 < \varepsilon' < \varepsilon$ by comparison with the geometric series. Inserting this formula for each variable, we conclude that f can be expanded in a power series with coefficients

$$a_{k_1 \ldots k_n} = \frac{1}{(2\pi i)^n} \int_{\Gamma_\varepsilon(p)} \frac{f(\zeta^1, \ldots, \zeta^n)}{(\zeta^n - p^n)^{k_n+1} \cdots (\zeta^1 - p^1)^{k_1+1}} \, d\zeta^1 \cdots d\zeta^n.$$

Assume (3) is true. Then Weierstrass' M-test implies that f is continuous, and complex variable theory implies that f has partial derivatives with respect to each z^j.

This completes the proof. □

Remark 1.8. *We could have alternatively defined $f : U \to \mathbb{C}$ to be holomorphic if and only if f admits a convergent power series about each point in U. Proposition 1.7 then implies that f is holomorphic in this new sense if and only if f is holomorphic in each variable separately. There is no analogue of this result for real variables. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function*

$$f(x,y) = \begin{cases} \frac{2x^2y+y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

This function is everywhere continuous and has well-defined partial derivatives with respect to x and y everywhere (including at the origin), but it is not differentiable at the origin. Indeed, we have,

$$\lim_{x\to 0} \frac{f(x,mx) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{2mx^3 + m^3x^3}{x(x^2 + m^2x^2)} = \lim_{x\to 0} \frac{x^3(2m + m^3)}{x^3(1 + m^2)} = \frac{2m + m^3}{1 + m^2}$$

Clearly, the limits are different for different values of m.

Remark 1.9. *As in one variable, we define the Wirtinger operators*

$$\frac{\partial}{\partial z^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right),$$
$$\frac{\partial}{\partial \overline{z^k}} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right).$$

An alternative definition is to say that a continuously differentiable function $f : U \to \mathbb{C}$ is holomorphic if it satisfies the Cauchy–Riemann equations

$$\frac{\partial f}{\partial \overline{z^k}} = 0 \text{ for } k = 1, 2, \ldots, n.$$

This follows readily from Proposition 1.10(3).

1.2. **Properties.** We now prove some properties of holomorphic functions of several variables that extend the properties of holomorphic functions of one variable.

Proposition 1.10. *Let $U \subseteq \mathbb{C}^n$ and let $f : U \to \mathbb{C}$ be a holomorphic function.*

(1) *If $g\colon U \to \mathbb{C}$ is a holomorphic function, then $f \pm g$ fg are holomorphic on U and f/g is holomorphic on $U \setminus g^{-1}(0)$.*
(2) *Let $W \subseteq \mathbb{C}^m$ be open. If $g\colon W \to U$ is a holomorphic, then $f \circ g$ is holomorphic.*
(3) *We have*

$$\frac{\partial f}{\partial z^j} = \frac{\partial f}{\partial x^j} = \frac{1}{i}\frac{\partial f}{\partial y^j}.$$

(4) *If $p \in U$ and $\Delta_\varepsilon(p) \subseteq U$, then the power series representation of f is given explicitly by the following formula*
$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} \frac{1}{k_1!\cdots k_n!}\frac{\partial^{k_1+\cdots+k_n} f(p)}{(\partial z^1)^{k_1}\cdots(\partial z^n)^{k_n}}(z^1-p^1)^{k_1}\cdots(z^n-p^n)^{k_n}.$$
(5) *(**Cauchy Estimate**) If $p \in U$ and $\Delta_\varepsilon(p) \subseteq U$, then*
$$\left|\frac{\partial^{k_1+\cdots+k_n} f(p)}{(\partial z^1)^{k_1}\cdots(\partial z^n)^{k_n}}\right| \le \|f\|_\infty \frac{k_1!\cdots k_n!}{\varepsilon^{k_1+\cdots+\varepsilon_n}},$$
where $\|f\|_\infty$ is the bounded on $|f|$ on $\Delta_\varepsilon(p)$.
(6) *(**Identity Theorem**) If U is connected and $g : U \to \mathbb{C}$ is anther holomorphic function that agrees with f on a non-empty open subset of $V \subseteq U$, then $f = g$ on U.*
(7) *(**Liouville's Theorem**) If $U = \mathbb{C}^n$ and f is bounded, then f is constant.*
(8) *(**Maximum Principle**) If $|f|$ attains a maximum value at some point in U, then f is constant.*
(9) *Let $f_k\colon U \to \mathbb{C}$ be a sequence of holomorphic functions that converge uniformly on compact subsets of U to a function $f\colon U \to \mathbb{C}$. Then f is holomorphic.*
(10) *(**Montel's Theorem**)*

Remark 1.11. *Recall that Proposition 1.10(7) is false for real-analytic functions.*

Proof. The proof is given below:
(1) This is clear.
(2) Certainly $f \circ g$ is continuous. Let $z = (z^1,\dots,z^m)$ denote the coordinates on W, and $w = (w^1,\dots,w^n)$ those on U. Then $w^j = g_j(z^1,\dots,z^m)$. By the chain rule, we have
$$\frac{\partial(f\circ g)}{\partial \bar z^k} = \sum_j \left(\frac{\partial g}{\partial w^j}\frac{\partial f_j}{\partial \bar z^k} + \frac{\partial g}{\partial \bar w^j}\frac{\partial \bar f_j}{\partial \bar z^k}\right) = 0,$$
This is zero because $\frac{\partial f_j}{\partial \bar z^k} = 0$ and $\frac{\partial g}{\partial \bar w^j} = 0$ by Remark 1.9. This is sufficient to infer that that $f \circ g$ is holomorphic.
(3) Compute the limits:
$$\frac{\partial f}{\partial z^j}(p) = \lim_{h\to 0,\, h\in\mathbb{R}} \frac{f(p^1,\dots,p^j+h,\dots,p^n) - f(p^1,\dots,p^n)}{h} = \frac{\partial f}{\partial x^j}(p),$$
$$\frac{\partial f}{\partial z^j}(p) = \lim_{k\to 0,\, k\in\mathbb{R}} \frac{f(p^1,\dots,p^j+ik,\dots,p^n) - f(p^1,\dots,p^n)}{ik} = \frac{1}{i}\frac{\partial f}{\partial y^j}(p).$$
Now simply note that the limits are equal.
(4) Simply differentiate the expression in Proposition 1.7(3) repeatedly term-by-term and evaluate at $z = p$ to determine the coefficients $a_{k_1,\cdots,k_n}$. This is justified by results concerning power series in several variables.
(5) Note that we have
$$\begin{aligned} a_{k_1\cdots k_n} &= \frac{1}{k_1!\cdots k_n!}\frac{\partial^{k_1+\cdots+k_n} f(p)}{(\partial z^1)^{k_1}\cdots(\partial z^n)^{k_n}} \\ &= \frac{1}{(2\pi i)^n}\int_{\Gamma_\varepsilon(p)} \frac{f(\zeta^1,\dots,\zeta^n)}{(\zeta^n-p^n)^{k_n+1}\cdots(\zeta^1-p^1)^{k_1+1}}\,d\zeta^1\cdots d\zeta^n. \end{aligned}$$

From the obvious bounds on the integrand of this integral, it follows that

$$|a_{k_1\cdots k_n}| \leq \frac{\|f\|_\infty}{\varepsilon^{k_1+\cdots+\varepsilon_n}}$$

The Cauchy estimate now follows.

(6) Set $h = f - g$, so $h \equiv 0$ on a nonempty open subset $V \subseteq U$. Let

$$W = \{z \in U \mid h \text{ and all its partial derivatives vanish at } z\}.$$

Then U is nonempty because $U_0 \subseteq U$. Let $z \in U$ be a limit point of W. There is a sequence of points $z^j \in W$ converging to z. Hence, all partial derivatives of h vanish at each z^j. By continuity, they also vanish at z. Hence, $z \in U$ implying that W is closed in U. Suppose $z \in W$. By (1) h is equal to a convergent power series in a neighborhood of z such that every term in the series is zero. Thus, W is open in U. Since W is cl-open and U is connected, the claim follows.

(7) Given any point $z \in \mathbb{C}^n$, define the function

$$g(\zeta) = f(\zeta z), \qquad \zeta \in \mathbb{C}$$

g is a bounded holomorphic function on $\mathbb{C}$. By Liouville's theorem from compelx variable theory, g is constant. Hence,

$$f(z) = g(1) = g(0) = f(0)$$

Since $z \in \mathbb{C}^n$ is arbitrary, f is constant.

(8) Suppose $|f|$ attains a maximum value at $z' \in U$. Consider the set,

$$W = \{z \in U \mid f(z) = f(z')\}.$$

Clearly, W is non-empty and closed. Given $z \in W$, choose $\varepsilon > 0$ such that $\Delta_\varepsilon(z) \subseteq U$. For each $w \in \mathbb{C}^n$ with $|w| = 1$, consider the function

$$g(\zeta) = f(z + \zeta w)$$

g is holomorphic on the disk $\mathbb{D}_\varepsilon(0) \subseteq \mathbb{C}$ and achieves its maximum modulus at $\zeta = 0$. By the maximum principle from complex variable theory, g is constant. Since w is arbitrary, this shows f is constant on $\Delta_\varepsilon(z)$. Thus, W is open. Since U is connected, $W = U$. Hence, $f \equiv f(z')$ on U.

(9) Given $p \in U$, choose $\varepsilon > 0$ such that $\overline{\Delta_\varepsilon}(p) \subset U$. For all $z \in \Delta_\varepsilon(p)$, we can apply the Cauchy integral formula to f_k, and uniform convergence guarantees that

$$\begin{aligned} f(z) &= \lim_{k\to\infty} \frac{1}{(2\pi i)^n} \int_{|\zeta^n - p^n| = r} \cdots \int_{|\zeta^1 - p^1| = r} \frac{f_k(\zeta^1, \ldots, \zeta^n)}{(\zeta^n - z^n)\cdots(\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n \\ &= \frac{1}{(2\pi i)^n} \int_{|\zeta^n - p^n| = r} \cdots \int_{|\zeta^1 - p^1| = r} \frac{f(\zeta^1, \ldots, \zeta^n)}{(\zeta^n - z^n)\cdots(\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n. \end{aligned}$$

Clearly, f is holomorphic.

(10)

This completes the proof. □

Remark 1.12. *Proposition 1.10 conveys that holomorphic functions are quite rigid. Indeed, here is an implication of Proposition 1.10. Let $f : \mathbb{C} \to \mathbb{H}$[1]. We claim that f is constant.*

[1] $\mathbb{H}$ is the upper half plane.

Note that $g(z) := e^{if(z)}$ is holomorphic on $\mathbb{C}$ with $\|g\|_\infty \le 1$[2]*. So g is a constant by Liouville's Theorem and hence f is a constant as well.*

So far, all these facts about holomorphic functions of several variables have been straightforward generalizations of standard facts about holomorphic functions of one variable. The next result, however, is radically different from anything in the one-variable theory.

Proposition 1.13. ***(Hartog's Extension Theorem)*** *Let $n \ge 2$, and let $U = \Delta_\varepsilon(p) \setminus \overline{\Delta_{\varepsilon'}(p)}$ for some $p \in \mathbb{C}^n$ and $0 < \varepsilon' < \varepsilon$. Every holomorphic function $f : U \to \mathbb{C}$ has a unique extension to a holomorphic function on all of $\Delta_\varepsilon(p)$.*

Proof. WLOG, we may assume that $p = 0$. Choose any $\delta > 0$ such that $\varepsilon' < \delta < \varepsilon$. As long as $\varepsilon' < |z^2| < \varepsilon$, the function $z^1 \mapsto f(z^1, \ldots, z^n)$ is holomorphic on $\Delta_\varepsilon(0) \subseteq \mathbb{C}$. Cauchy's integral formula shows that

$$f(z^1, \ldots, z^n) = \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{f(\zeta, z^2, \ldots, z^n)}{\zeta - z^1}\, d\zeta.$$

This formula actually makes sense for all $(z^1, \ldots, z^n) \in \Delta_\delta(0)$ because the integration contour is contained in U and it defines a holomorphic function f_1 there by differentiation under the integral sign. Because f_1 agrees with f on the open subset of $\Delta_\delta(0)$ where $\varepsilon' < |z^2| < \delta$, the identity theorem shows that it agrees on the entire connected set $\Delta_\delta(0) \setminus \Delta_{\varepsilon'}(0)$. Thus we can define a holomorphic function on all of $\Delta_\varepsilon(0)$ by letting it be equal to f on U and to f_1 on $\Delta_\delta(0)$. Uniqueness follows from the identity theorem. □

Remark 1.14. *In the language of sheaves of holomorphic functions, Proposition 1.13 states that the map $\mathscr{O}_{\mathbb{C}^n}(\Delta_\varepsilon(p)) \to \mathscr{O}(\Delta_\varepsilon(p) \setminus \Delta_{\varepsilon'}(p))$ is bijective.*

Remark 1.15. *Proposition 1.13 is false if $n = 1$. Let $f(z) = 1/z$ on an annular region centered at the origin. Then f is holomorphic on the annular region but f is not holomorphic on large unit disk defining the annular region.*

Proposition 1.13 implies that singularities of holomorphic functions in two or more variables are never isolated. Similarly, it implies that the zeros of a holomorphic function in two or more variables are never isolated. If not, then let f have an isolated zero at p. Then $1/f$ would have an isolated singularity, which is a contradiction.

Remark 1.16. *A holomorphic function of one complex variable may have isolated singularities. Simply consider $f(z) = 1/z$ on $\mathbb{C}$.*

Remark 1.17. *The zero of a holomorphic function of one complex variable are always isolated. The claim is obviously true if $f \in \mathbb{C}[z]$. Generally, if f is a holomorphic function on an open set and f has a zero of multiplicity k at $a \in U$, then we have*

$$f(z) = \sum_{i=k}^{\infty} \frac{f^{(i)}(a)}{i!}(z-a)^i = (z-a)^k \sum_{i=k}^{\infty} \frac{f^{(i)}(a)}{i!}(z-a)^{i-k} := (z-a)^k g(z), \quad 0 \le |z-a| < r$$

for some $r \in \mathbb{R}$ such that the open disk is contained in U. Since $g \ne 0$ on $0 < |z-a| < r$, we have that $f \ne 0$ on $0 < |z-a| < r$.

We end this section by proving the Schwarz lemma:

[2]If $f(z) = a_z + ib_z$, then $e^{if(z)} = e^{ia_z - b_z}$. Hence $|e^{if(z)}| = e^{-b_z} < 1$ since $b_z > 0$.

Proposition 1.18. ***(Schwarz Lemma)*** *Let $\Delta_\varepsilon(0)$ be a polydisc and let $f : U \to \mathbb{C}$ be a holomorphic function such that $\overline{\Delta_\varepsilon(0)} \subseteq U$. Assume that f non-trivial monomials of degree $< k$ do not occur in the power series expansion of f. If $|f(z)| \leq C$ on $\overline{\Delta_\varepsilon(0)}$ can be bounded from then*

$$|f(z)| \leq C|z|^k\varepsilon^{-k}$$

for all $z \in \overline{\Delta_\varepsilon(0)}$.

Proof. Let $\mathbb{D}_\varepsilon(0)$ be a unit ball in $\mathbb{C}$. Fix $0 \neq z \in \Delta_\varepsilon(0)$. Define

$$g_z(w) = w^{-k}f(wz/|z|), \quad w \in \mathbb{D}_\varepsilon(0).$$

Then $|g_z(w)| \leq C\varepsilon'^{-k}$ for $|w| = \varepsilon' < \varepsilon$. The maximum principle implies that $|g_z(w)| \leq C\varepsilon'^{-k}$ for $w \in \mathbb{D}_{\varepsilon'}(0)$. If $|z| = \varepsilon' < \varepsilon$, we have,

$$|z|^{-k}|f(z)| = |g_z(|z|)| \leq C\varepsilon'^{-k}$$

Since $\varepsilon' < \varepsilon$ is arbitrary, we have

$$|z|^{-k}|f(z)| = |g_z(|z|)| \leq C\varepsilon^{-k}$$

This completes the proof. □

2. Complexification & Complex Structures

2.1. **Complexification.** We begin by discussing the technique of complexification, which allows us to *complexify* a $\mathbb{R}$-vector space.

Definition 2.1. If V is a $\mathbb{R}$-vector space, we define the **complexification** of V, denoted by $V^{\mathbb{C}}$, to be the $\mathbb{C}$-vector space $V \oplus V$ with scalar multiplication by complex numbers defined as follows:

$$(a + ib)(u, v) = (au - bv, av + bu) \quad \text{for } a + ib \in \mathbb{C}.$$

Remark 2.2. *$V^{\mathbb{C}}$ is then a $\mathbb{C}$-vector space over $\mathbb{C}$.*

The map $V \to V^{\mathbb{C}}$ given by $u \mapsto (u, 0)$ is a $\mathbb{R}$-linear isomorphism from V onto the (real) subspace $V \oplus \{0\} \subseteq V^{\mathbb{C}}$. We identify V with its image under this map. We can write

$$(u, v) = u + iv$$

and we can think of $V^{\mathbb{C}}$ as consisting of the set of all linear combinations of elements of V with complex coefficients. If $\dim_{\mathbb{R}} V = n$ and $\{b_1, \ldots, b_n\}$ is any basis for V (over $\mathbb{R}$), then

$$\{(b_1, 0), \ldots, (b_n, 0)\}$$

is a basis for $V^{\mathbb{C}}$ over $\mathbb{C}$. Hence $\dim_{\mathbb{C}} V^{\mathbb{C}} = n$. On the other hand,

$$\{(b_1, 0), \ldots, (b_n, 0), (0, ib_1), \ldots, (0, ib_n)\}$$

is a basis for $V^{\mathbb{C}}$ over $\mathbb{R}$. Hence $\dim_{\mathbb{R}} V^{\mathbb{C}} = 2n$.

Definition 2.3. If $L\colon V \to W$ is a linear map between $\mathbb{R}$-vector spaces, the **complexification** of L is the $\mathbb{C}$-linear map

$$\begin{aligned} L^{\mathbb{C}}\colon V^{\mathbb{C}} &\to W^{\mathbb{C}}, \\ u + iv &\mapsto L(u) + iL(v) \end{aligned}$$

For $k = \mathbb{R}, \mathbb{C}$ if $\mathbf{Vec}_k$ is the category of finite-dimensional k-vector spaces, then complexification can be thought of as a functor,

$$\mathscr{F} : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Vec}_{\mathbb{C}}$$

such that $\mathscr{F}(V) = V^{\mathbb{C}}$ and $\mathscr{F}(L) = L^{\mathbb{C}}$. Clearly, $\mathscr{F}(\mathrm{Id}_V) = \mathrm{Id}_{V^{\mathbb{C}}}$. Moreover if $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ are $\mathbb{R}$-linear maps, then

$$\begin{aligned}
\mathscr{F}(L_2 \circ L_1)(u + iv) &= (L_2 \circ L_1)^{\mathbb{C}}(u + iv) \\
&= L_2(L_1(u)) + iL_2(L_1(v)) \\
&= (L_2^{\mathbb{C}} \circ L_1^{\mathbb{C}})(u + iv) \\
&= (\mathscr{F}(L_2) \circ \mathscr{F}(L_1))(u + iv)
\end{aligned}$$

Hence, $\mathscr{F}(L_2 \circ L_1) = \mathscr{F}(L_2) \circ \mathscr{F}(L_1)$.

Remark 2.4. *There is another way to think about complexification. If V is a $\mathbb{R}$-vector space, we can consider the $(V')^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. $(V')^{\mathbb{C}}$ is a $\mathbb{C}$-vector space with the usual addition and with scalar multiplication defined by*

$$\alpha(v \otimes \beta) = v \otimes (\alpha\beta), \qquad v \in V \ \alpha, \beta \in \mathbb{C}$$

Consider the map

$$\begin{aligned}
\phi \colon V^{\mathbb{C}} &\to (V')^{\mathbb{C}} \\
(u, v) &\mapsto u \otimes_{\mathbb{R}} 1 + v \otimes_{\mathbb{R}} i
\end{aligned}$$

The map is $\mathbb{C}$-linear. Indeed, ϕ is additive because $\otimes_{\mathbb{R}}$ is bilinear. Moreover, we have,

$$\begin{aligned}
\phi((a + ib)(u, v)) &= \phi(au - bv, av + bu) \\
&= (au - bv) \otimes_{\mathbb{R}} 1 + (av + bu) \otimes_{\mathbb{R}} i \\
&= u \otimes_{\mathbb{R}} (a + ib) + v \otimes_{\mathbb{R}} i(a + ib) \\
&= (a + ib)(u \otimes_{\mathbb{R}} 1 + v \otimes_{\mathbb{R}} i) \\
&= (a + ib)\phi(u, v)
\end{aligned}$$

Hence, ϕ is $\mathbb{C}$-linear. ϕ is surjective. Indeed if $v \otimes (a + ib) \in (V')^{\mathbb{C}}$, then

$$v \otimes_{\mathbb{R}} (a + ib) = av \otimes_{\mathbb{R}} 1 + bv \otimes_{\mathbb{R}} i$$

implies that $\phi(av, bv) = v \otimes (a + ib)$. Since $V^{\mathbb{C}}$ and $(V')^{\mathbb{C}}$ are finite-dimensional $\mathbb{C}$-vector spaces, ϕ is a $\mathbb{C}$-linear isomorphism.

2.2. **Complex Structures.** We now discuss complex structures. A complex structure is a property of a $\mathbb{R}$-vector space that allows it to be treated as a $\mathbb{C}$-vector space. To motivate this, consider the following: Let V be a $\mathbb{C}$-vector space. Scalar multiplication by i defines a linear map $v \mapsto iv$ on the underlying $\mathbb{R}$-vector space, which squares to $-I$. By ignoring the $\mathbb{C}$-vector space structure, we can think of J as a $\mathbb{R}$-linear map satisfying $J \circ J = -\mathrm{Id}$.

Definition 2.5. Let V be a $\mathbb{R}$-vector space. A **complex structure** on V is a $\mathbb{R}$-linear map $J \colon V \to V$ satisfying $J \circ J = -\mathrm{Id}_V$.

Example 2.6. Let $V = \mathbb{C}^n$. Then

$$\mathbb{C}^n \cong \mathbb{R}^{2n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid x_i, y_i \in \mathbb{R}\}$$

and the complex structure $J^{\mathbb{C}^n}$ is given by

$$J^{\mathbb{C}^n}(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-y_1, \ldots, -y_n, x_1, \ldots, x_n).$$

Lemma 2.7. *Let V be a $\mathbb{R}$-vector space. If J is a complex structure on V, then V admits in a natural way the structure of a $\mathbb{C}$-vector space.*

Proof. Define scalar multiplication by $\mathbb{C}$ on V by

$$(a+ib)v = av + bJ(v),$$

where $a, b \in \mathbb{R}$. The $\mathbb{R}$-linearity of J and the assumption $J^2 = -\mathrm{Id}$ yield

$$((a+ib)(c+id))v = (a+ib)((c+id)\cdot v),$$

This completes the proof. $\square$

Remark 2.8. *If V is a $\mathbb{R}$-linear vector space admitting a complex structure, J, then the $\mathbb{C}$-linear vector space structure on V is denoted as (V, J). Combined with our discussion of complexification, we see that complex structures on $\mathbb{R}$-vector spaces and $\mathbb{C}$-vector spaces are equivalent notions.*

Proposition 2.9. *Let V be a $\mathbb{R}$-vector space. If J is a complex structure V, then $V^{\mathbb{C}}$ has an eigenspace decomposition of the form*

$$\begin{aligned} V^{\mathbb{C}} &= V_{(1,0)} \oplus V_{(0,1)} \\ &= \left\{ \frac{v - iJv}{2} \mid v \in V \right\} \bigoplus \left\{ \frac{v + iJv}{2} \mid v \in V \right\}. \end{aligned}$$

where $V_{1,0}$ is the i-eigenspace of $J^{\mathbb{C}}$ and $V_{0,1}$ is the $-i$-eigenspace of J. Moreover, $V_{(1,0)} \cong V_{(0,1)}$ as $\mathbb{R}$-linear vector spaces.

Proof. Given $v \in V^{\mathbb{C}}$, define v' and v'' by the formulas

$$v' = \frac{1}{2}(v - iJv), \quad w'' = \frac{1}{2}(v + iJv).$$

A simple computation show that

$$J^{\mathbb{C}}v' = iv' \quad J^{\mathbb{C}}v'' = -iv''$$

Because $v = v' + v''$, this shows that $V^{\mathbb{C}} = V_{(1,0)} + V_{(0,1)}$. A non-zero vector cannot be an eigenvector with two different eigenvalues, so $V_{(1,0)} \cap V_{(0,1)} = \{0\}$, which shows that the sum is direct. Conjugation is a bijective $\mathbb{R}-$linear map from $V^{\mathbb{C}}$ to itself and it interchanges $V_{(1,0)}$ and $V_{(0,1)}$. This shows that $V_{(1,0)} \cong V_{(0,1)}$ as $\mathbb{R}$-linear vector spaces. $\square$

Lemma 2.10. *Let V be a finite-dimensional $\mathbb{R}$-vector space. If V admits a complex structure, then $\dim_{\mathbb{R}} V$ is even.*

Proof. If V admits a complex structure, then Proposition 2.9 implies that $\dim_{\mathbb{R}} V_{(1,0)} = \dim_{\mathbb{R}} V_{(0,1)}$. In turn, this implies that $\dim_{\mathbb{C}} V_{(1,0)} = \dim_{\mathbb{C}} V_{(0,1)}$. This follows because since $V_{(1,0)}, V_{(0,1)}$ are $\mathbb{C}$-vector spaces, we have that $\dim_{\mathbb{C}} V_{(1,0)}, V_{(0,1)}$ is half that of $\dim_{\mathbb{R}} V_{(1,0)}, V_{(0,1)}$. Hence, $\dim_{\mathbb{C}} V^{\mathbb{C}}$ is even. Since $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$, the result follows. $\square$

We close this section with an important observation that will be useful later on:

Proposition 2.11. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. Then*

(1) *We have*

$$(V^{\mathbb{C}})^* = (V^*)^{\mathbb{C}}.$$

(2) *We have*

$$(V^*)^{\mathbb{C}} \cong ((V^*, J^*))_{(1,0)} \oplus ((V^*, J^*))^*_{(0,1)}$$

Here J^* is a complex structure on V^* we view (V^*, J^*) as a $\mathbb{C}$-linear space via Lemma 2.7.

Proof. The proof is given below:

(1) Note that we have

$$\begin{aligned}(V^{\mathbb{C}})^* &= \mathrm{Hom}_{\mathbb{C}}(V^{\mathbb{C}}, \mathbb{C}) \\ &= \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathbb{R}}(V, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})) \\ &\cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = (V^*)^{\mathbb{C}}\end{aligned}$$

The second isomorphism follows from the tensor-hom adjunction since V is a $(\mathbb{R}, \mathbb{C})$-vector space. The isomorphism $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ is clear.

(2) First note that V^* has a complex structure given by the $\mathbb{R}$-linear map

$$\begin{aligned}J^* : V^* &\to V^*, \\ \varphi &\mapsto \varphi \circ J\end{aligned}$$

The isomorphism

$$(V^*)^{\mathbb{C}} \cong (V_{(1,0)})^* \oplus (V_{(0,1)})^*$$

follows from (1) and Proposition 2.9. It suffices to prove that $(V_{(1,0)})^* \cong (V^*, J^*)_{(1,0)}$. Note that there exists a $\mathbb{C}$-linear isomorphism

$$\begin{aligned}(V, J) &\to V_{(1,0)}, \\ v &\mapsto \frac{1}{2}(v - iJv).\end{aligned}$$

Here V is viewed as a $\mathbb{C}$-linear space via Lemma 2.7. This induces the $\mathbb{C}$-linear isomorphism $\mathrm{Hom}_{\mathbb{C}}(V_{(1,0)}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}((V, J), \mathbb{C})$. Therefore, we have

$$\begin{aligned}(V^*, J^*)_{(1,0)} &= \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid J^*(\varphi) = i\varphi\} \\ &= \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid \varphi \circ J = i\varphi\} \\ &\simeq \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(V_{(1,0)}, \mathbb{C}) = (V_{(1,0)})^*.\end{aligned}$$

This completes the proof. □

3. Multilinear & Alternating Maps

3.1. **Multilinear Maps.** We first discuss multi-linear maps.

Definition 3.1. For $i = 1, \cdots, k$, let V_i be $\mathbb{R}$-vector spaces, and let W be any other $\mathbb{R}$-vector space. A map

$$\omega\colon \underbrace{V_1 \times \cdots \times V_k}_{k-\text{times}} \to \mathbb{R}$$

is said to be k-multi-linear if it is linear as a function of each variable separately when the others are held fixed. For each i, this means:

$$\omega(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = a\omega(v_1, \ldots, v_i, \ldots, v_k) + a'\omega(v_1, \ldots, v_i', \ldots, v_k),$$

where $a, a' \in \mathbb{R}$ and $v_i, v_i' \in V_i$.

Remark 3.2. *It can be easily checked that the set of k-multi-linear maps is a $\mathbb{R}$-linear space, which we denote as $L(V_1, \ldots, V_k; W)$. A k-multi-linear map is also called a k-tensor.*

When $W = \mathbb{R}$, we can characterize the vector space of k-multi-linear maps as follows:

Proposition 3.3. *For $i = 1, \cdots, k$, let V_i be $\mathbb{R}$-vector spaces such that $\dim_{\mathbb{R}} = n_i$. There is a canonical isomorphism:*

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R}),$$

Hence,

$$\dim_{\mathbb{R}} L(V_1, \ldots, V_k; \mathbb{R}) = n_1 \cdots n_k$$

Proof. Define a map $\Phi \colon V_1 \times \cdots \times V_k \to L(V_1, \ldots, V_k; \mathbb{R})$ such that

$$\Phi(\omega^1, \ldots, \omega^k)(v_1, \ldots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k).$$

The expression on the right depends linearly on each v_i, so $\Phi(\omega_1, \ldots, \omega_k)$ is indeed an element of the space $L(V_1, \ldots, V_k; \mathbb{R})$. It is easy to check that Φ is k-multi-linear as a function of $\omega_1, \ldots, \omega_k$. By the characteristic property on tensor products, it descends uniquely to a linear map $\overline{\Phi} : V_1^* \otimes \cdots \otimes V_k^* \to L(V_1, \ldots, V_k; \mathbb{R})$, which satisfies

$$\overline{\Phi}(\omega_1 \otimes \cdots \otimes \omega_k)(v_1, \ldots, v_k) = \omega_1(v_1) \cdots \omega_k(v_k).$$

The linear map $\overline{\Phi}$ takes the basis of $V_1^* \otimes \cdots \otimes V_k^*$ to the basis for $L(V_1, \ldots, V_k; \mathbb{R})$. So it is an isomorphism. $\square$

Remark 3.4. *From now on, assume that $V_i = V$, $\dim_{\mathbb{R}} V = n$ for each i and $W = \mathbb{R}$.*

3.2. **Alternating Maps.** We now specialize to the case of alternating k-multi-linear maps.

Definition 3.5. A map

$$\omega \colon \underbrace{V \times \cdots \times V}_{k-\text{times}} \to \mathbb{R}$$

is said to be an alternating k-multi-linear map if it is k-multi-linear and that for every pair of distinct indices i, j, it satisfies

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Remark 3.6. *If ω is an alternating k-multi-linear map, then the effect of an arbitrary permutation $\sigma \in S_k$ of its arguments is given by*

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\omega(v_1, \ldots, v_k),$$

where $\operatorname{sgn}(\sigma)$ represents the sign of the permutation $\sigma \in S_k$. This follows from repeated applications of the definition of an alternating k-multi-linear map and the definition of $\operatorname{sgn}(\sigma)$ for $\sigma \in S_k$.

Remark 3.7. *It can be easily checked that the set of alternating k-multi-linear maps is a $\mathbb{R}$-linear space, which we denote as $A(V_1, \ldots, V_k; W)$. An alternating k-multi-linear map is also called a k-form.*

Example 3.8. The determinant is a $\mathbb{R}$-valued multilinear n-form in $\mathbb{R}^n$. That is, the map $\det \colon (\mathbb{R}^n)^n \to \mathbb{R}$ is a an alternating n-multi-linear map.

The following lemma gives a different alternative characterization for the alternating condition:

Lemma 3.9. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. Let ω be a k-multi-linear map. The following are equivalent:*

(1) *ω is an alternating k-multi-linear map.*
(2) *$\omega(v_1, \dots, v_k) = 0$ whenever the k-tuple $(v_1, \dots, v_k)$ is linearly dependent.*
(3) *ω gives the value zero whenever two of its arguments are equal:*

$$\omega(v_1, \dots, w, \dots, w, \dots, v_k) = 0.$$

Proof. (1),(2) imply (3) are immediate. We complete the proof by showing that (3) implies both (1) and (2). Assume that ω satisfies (3). For any vectors $v_1, \dots, v_k$, the hypothesis implies

$$\begin{aligned}
0 &= \omega(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) \\
&= \omega(v_1, \dots, v_i, \dots, v_i, \dots, v_k) + \omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\
&\quad + \omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k) + \omega(v_1, \dots, v_j, \dots, v_j, \dots, v_k) \\
&= \omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + \omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k).
\end{aligned}$$

Thus, ω is an alternating k-multi-linear map. Hence, (3) implies (1). If $(v_1, \dots, v_k)$ is a linearly dependent k-tuple, then one of the v_i's can be written as a linear combination of the others. For simplicity, let us assume that $v_k = \sum_{j=1}^{k-1} a_j v_j$. Since ω is k-multi-linear, we have,

$$\omega(v_1, \dots, v_k) = \sum_{j=1}^{k-1} a_j \omega(v_1, \dots, v_{k-1}, v_j).$$

In each of these terms, ω has two identical arguments, so every term is zero. Hence, (3) implies (2). $\square$

Remark 3.10. *In what follows, let $\{E_i\}_{i=1}^n$ denote a basis for V, and let $\{\varepsilon^j\}_{j=1}^n$ denote the dual basis for V^*.*

Note that Proposition 3.3 implies that the set

$$\{\varepsilon^{i_1} \otimes \cdots \varepsilon^{i_k} \mid 1 \leq i_1, \cdots, i_k, \leq n\}$$

is a basis for $L(V, \cdots, V; \mathbb{R})$. We would like to find a basis for $A(V, \cdots, V; \mathbb{R})$, the subspace of alternating k-multi-linear maps. Preempting the discussion in the next section, we write $A(V, \cdots, V; \mathbb{R})$ as $\Lambda^k(V)$ in the remainder of this section. We first define a collection of alternating k-multi-linear maps on V that generalize the determinant function on $\mathbb{R}^n$.

Definition 3.11. Let V be a $\mathbb{R}$-vector space and let $\dim_{\mathbb{R}} V = n$. For each multi-index $I = (i_1, \dots, i_k)$ of length k such that $1 \leq i_1, \dots, i_k \leq n$, define an alternating k-multi-linear map as follows:

$$\begin{aligned}
\varepsilon^I : \underbrace{V \times \cdots \times V}_{k\text{-times}} &\to \mathbb{R} \\
(v_1, \dots, v_k) &\mapsto \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{pmatrix}
\end{aligned}$$

Because the determinant changes sign whenever two columns are interchanged, it is clear that ε^I in Definition 3.11 is an alternating k-multi-linear map.

Remark 3.12. *We introduce some notation. If $I = (i_1, \cdots, i_k)$ is a multi-index and $\sigma \in S_k$ is a permutation of $\{1, \ldots, k\}$, we write I_σ for the following multi-index:*

$$I_\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(k)}).$$

Note that $I_{\sigma\tau} = (I_\sigma)_\tau$ for $\sigma, \tau \in S_k$. If I and J are multi-indices of length k, we define δ^I_J as follows:

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_k} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_k} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \delta^{i_k}_{j_2} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}$$

We have

$$\delta^J_I = \begin{cases} \operatorname{sgn}(\sigma) & \textit{if neither } I \textit{ nor } J \textit{ has a repeated index and } J = I_\sigma \textit{ for some } \sigma \in S_k, \\ 0 & \textit{if } I \textit{ or } J \textit{ has a repeated index or } J \textit{ is not a permutation of } I. \end{cases}$$

Lemma 3.13. *The following statements are true:*

(1) *If I has a repeated index, then $\varepsilon^I = 0$.*
(2) *If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = sgn(\sigma)\varepsilon^J$.*
(3) *The result of evaluating ε^I on a sequence of basis vectors is*

$$\varepsilon^I(E_{j_1}, \ldots, E_{j_k}) = \delta^I_J$$

Proof. If I has a repeated index, then for any vectors $v_1, \ldots, v_k$, the determinant in the definition of ε^I has two identical rows and thus is equal to zero, which proves (1). On the other hand, if J is obtained from I by interchanging two indices, then the corresponding determinants have opposite signs; this implies (2). Finally, (3) follows immediately from the definition of ε^I. $\square$

The importance of the k-alternating tensors ε^I, for an increasing multi-index I, is given by the following proposition:

Proposition 3.14. *For each positive integer $0 \leq k \leq n$, the collection of k-alternating tensors*

$$\mathcal{E} = \{\varepsilon^I : I \textit{ is an increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Therefore,

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n$, then $dim\,\Lambda^k(V^*) = 0$.

Remark 3.15. *A multi-index $I = (i_1, \ldots, i_k)$ is said to be increasing if $i_1 < \ldots < i_k$.*

Proof. The fact that $\Lambda^k(V^*)$ is the trivial vector space when $k > n$ follows immediately from Lemma 3.13, since every k-tuple of vectors is linearly dependent in that case. So let $k \leq n$. To show that $\mathcal{E}$ spans $\Lambda^k(V^*)$, let $\alpha \in \Lambda^k(V^*)$ be arbitrary. For each multi-index $I = (i_1, \ldots, i_k)$ (not necessarily increasing), define a real number α_I by

$$\alpha_I = \alpha(E_{i_1}, \ldots, E_{i_k}).$$

Lemma 3.13 implies:

$$\sum_I \alpha_I \varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \sum_I \alpha_I \delta^I_J = \alpha_J = \alpha(E_{j_1}, \dots, E_{j_k}).$$

Thus, $\sum_I \alpha_I \varepsilon^I = \alpha$, so $\mathcal{E}$ spans $\Lambda^k(V^*)$. To show that $\mathcal{E}$ is a linearly independent set, suppose the identity

$$\sum_I \alpha_I \varepsilon^I = 0$$

holds for some coefficients α_I. Let J be any increasing multi-index. Applying both sides of the identity to the vectors $E_{j_1}, \dots, E_{j_k}$ and using Lemma 3.13, we get

$$0 = \sum_I \alpha_I \varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \alpha_J.$$

Thus, each coefficient α_J is zero. □

3.3. **Wedge Product.** We can go a step further and define a product operation for alternating k-multi-linear maps. Recall that there is a product operation on the $\mathbb{R}$-linear space of k-multi-linear maps. If

$$\omega : \underbrace{V \times \cdots \times V}_{k-\text{times}} \to \mathbb{R}$$

$$\gamma : \underbrace{V \times \cdots \times V}_{l-\text{times}} \to \mathbb{R}$$

multi-linear maps we can define their product $\omega \otimes \gamma$ to be a $(k+l)k$-multi-linear map

$$\omega \otimes \gamma : \underbrace{V \times \cdots \times V}_{(k+l)-\text{times}} \to \mathbb{R}$$

such that

$$\omega \otimes \gamma(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = \omega(v_1, \dots, v_k)\gamma(v_{k+1}, \dots, v_{k+l})$$

Now consider the case of alternating multi-linear maps. Given $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^l(V^*)$, we define their wedge product (or exterior product) to be the following element in $\Lambda^{k+l}(V^*)$:

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta) =$$

If ω is a k-multi-linear map, then $\operatorname{Alt}(\omega)$ is defined as

$$\operatorname{Alt}(\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

It is easy to verify that Alt defines a projection operator:

$$\operatorname{Alt} : L(V, \cdots, V; \mathbb{R}) \to A(V, \cdots, V; \mathbb{R})$$

The mysterious coefficient is motivated by statement of Lemma 3.16:

Lemma 3.16. *For any multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$,*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ},$$

where $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$ is obtained by concatenating I and J.

Proof. By multilinearity, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J \left(E_{p_1}, \cdots, E_{p_{k+1}}\right) = \varepsilon^{IJ} \left(E_{p_1}, \cdots, E_{p_{k+1}}\right)$$

We consider several cases.

- If $P = (p_1, \ldots, p_k, q_1, \ldots, q_l)$ has a repeated index. In this case, both sides above of are zero.
- If P contains an index that does not appear in either I or J, the right-hand side is zero. Similarly, each term in the expansion of the left-hand side involves either I or J evaluated on a sequence of basis vectors that is not a permutation of I or J, respectively, so the left-hand side is also zero.
- If $P = IJ$ and P has no repeated indices, the right-hand side of is equal to 1. So we need to show that the left-hand side is also equal to 1. By definition,

$$\varepsilon^I \wedge \varepsilon^J \ \left(E_{p_1}, \cdots, E_{p_{k+1}}\right) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \varepsilon^I \left(E_{\sigma(1)}, \cdots, E_{\sigma(k)}\right) \varepsilon^J \left(E_{\sigma(k+1)}, \cdots, E_{\sigma(k+l)}\right).$$

  By Lemma 3.13, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last l indices of P separately. In other words, σ must be of the form $\sigma = \tau\eta$, where $\tau \in S_k$ acts by permuting $\{1, \ldots, k\}$ and $\eta \in S_l$ acts by permuting $\{k+1, \ldots, k+l\}$. Since $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\eta)$, we have:

$$\begin{aligned}\varepsilon^I \wedge \varepsilon^J \left(E_{p_1}, \cdots E_{p_{k+l}}\right) &= \frac{1}{k!l!} \sum_{\tau \in S_k\ \eta \in S_l} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\, \varepsilon^I \left(E_{p_{\tau(1)}}, \cdots, E_{p_{\tau(k)}}\right) \varepsilon^J \left(E_{p_{k+\eta(1)}}, \cdots, E_{p_{k+\eta(l)}}\right) \\ &= \varepsilon^I(E_{p_1}, \cdots E_{p_k})\varepsilon^J(E_{p_{k+1}}, \cdots E_{p_{k+l}}) = 1\end{aligned}$$

- If P is a permutation of IJ and has no repeated indices, applying a permutation to P brings us back to the above case. Since the effect of the permutation is to multiply both sides of

$$\varepsilon^I \wedge \varepsilon^J \left(E_{p_1}, \cdots, E_{p_{k+1}}\right) = \varepsilon^{IJ} \left(E_{p_1}, \cdots, E_{p_{k+1}}\right)$$

  by the same sign, the result holds in this case as well.

This completes the proof. □

Proposition 3.17. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. Let α, β, γ be alternating k-multi-linear maps on V. The wedge product satisfies the following properties:*

(1) *(Bilinearlity) For $a, b \in \mathbb{R}$,*

$$(a\alpha + b\beta) \wedge \gamma = a\alpha \wedge \gamma + b\beta \wedge \gamma$$

(2) *(Associativity)*

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

(3) *(Graded Anti-Commutivity) If $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^l(V^*)$,*

$$\alpha \wedge \beta = (-1)^{kl}\, \beta \wedge \alpha$$

(4) *If $I = (i_1, \ldots, i_k)$ is any multi-index, then*

$$\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} = \varepsilon^I$$

Proof. (1) follows immediately from the definition, because the tensor product is bilinear and Alt is linear. By Lemma 3.16,

$$(\varepsilon^I \wedge \varepsilon^J) \wedge \varepsilon^K = \varepsilon^{IJ} \wedge \varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^I \wedge (\varepsilon^J \wedge \varepsilon^K)$$

The general case follows from bilinearity. Similarly, using Lemma 3.16 again, we get

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = \operatorname{sgn}(\sigma)\, \varepsilon^{JI} = \operatorname{sgn}(\sigma)\, \varepsilon^J \wedge \varepsilon^I,$$

where σ is the permutation that sends IJ to JI. It is easy to check that $\operatorname{sgn}(\sigma) = (-1)^{kl}$, because σ can be decomposed as a composition of kl transpositions (each index of I must be moved past each of the indices of J). (3) then follows from bilinearity. (4) is an immediate consequence of Lemma 3.13. □

Proposition 3.18. *The wedge product is the unique associative, bilinear map*

$$\Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$$

satisfying (1)-(4) in Proposition 3.17.

Proof. If we take $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ then we can expand in the usual basis as $\omega = \sum_I \omega_I \varepsilon^I$, $\eta = \sum_J \eta_J \varepsilon^J$. Taking $*$ to be a map satisfying these four properties. By bilinearity,

$$\omega * \eta = \left(\sum_I \omega_I \varepsilon^I\right) * \left(\sum_J \eta_J \varepsilon^J\right) = \sum_I \sum_J \omega_I \eta_J \left(\varepsilon^I * \varepsilon^J\right)$$

Using associativity and (4) in Proposition 3.17

$$\varepsilon^I * \varepsilon^J = (\varepsilon^{i_1} * \cdots * \varepsilon^{i_k}) * (\varepsilon^{j_1} * \cdots * \varepsilon^{j_l}) = \varepsilon^{i_1} * \cdots * \varepsilon^{i_k} * \varepsilon^{j_1} * \cdots * \varepsilon^{j_l} = \varepsilon^{IJ} = \varepsilon^I \wedge \varepsilon^J$$

This proves the claim. □

Remark 3.19. *We can now define a $\mathbb{R}$-vector space $\Lambda(V)$*

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*),$$

Clearly, $\dim_{\mathbb{R}} \Lambda(V) = 2^n$*. The wedge product turns $\Lambda(V)$ into an associative algebra, called the exterior algebra of V^*. This algebra is not commutative, but it is graded-anticommutative in the sense that if $\alpha \in \Lambda^k(V^*), \beta \in \Lambda^l(V^*)$, then $\alpha \wedge \beta \in \Lambda^{k+l}(V^*)$ and*

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

4. Exterior Algebra

4.1. **Definitions.** Let V be an $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. In the previous section we have constructed a graded associative algebra, $\Lambda(V^*)$, such that if $\alpha \in \Lambda(V^*)$, then $\alpha \wedge \alpha = 0$. Since V is finite-dimensional, we can identify V with $(V^*)^*$. Hence, we formally have

$$\Lambda(V) = \Lambda((V^*)^*) = \bigoplus_{k=0}^{n} \Lambda^k((V^*)^*)$$

Exploiting Proposition 3.3 If an element $\omega \in \Lambda((V^*)^*)$ is an alternating k-multi-linear map that can be identified with an element of $\underbrace{V \otimes \cdots \otimes V}_{k-\text{times}}$, we must have that $\omega \otimes \omega$ under this identification. This motivates the following definition:

Definition 4.1. Let V be a $\mathbb{R}$-vector space. Consider the tensor algebra $T(V)$:

$$T(V) = \bigoplus_{k=0}^{\infty} T^k V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

The **exterior algebra** of V is the quotient algebra $\overline{\Lambda}(V) = T(V)/I$, where I is the two-sided ideal generated by all elements of the form $v \otimes v$ such that $v \in V$.

In analogy with the definition of the wedge product, we write an arbitrary element in $\overline{\Lambda}(V)$ is written as

$$v_1 \overline{\wedge} \cdots \overline{\wedge} v_k := \overline{v_1 \otimes \cdots \otimes v_k}$$

for some $k \geq 0$. Moreover, in analogy with the analogous construction in the previous section, we can define the following subspace of $\overline{\Lambda}(V)$:

$$\overline{\Lambda}^k(V) = \mathrm{Span}_{\mathbb{R}} = \{v_1 \overline{\wedge} \cdots \overline{\wedge} v_k \mid 1, i_1, \cdots, i_k \leq n\} \subseteq \Lambda(V)$$

We have the following basic properties:

Lemma 4.2. *Let V be a $\mathbb{R}$-vector space.*

(1) *We have $v \overline{\wedge} w = -w \overline{\wedge} v$ for each $v, w \in V$.*
(2) *For $k > n$, $\overline{\Lambda}^k(V) = 0$.*

Proof. The proof is given below:

(1) By construction $v \overline{\wedge} v = 0 \quad$ for all $v \in V$. Consequently, if $v, w \in V$ we have

$$\begin{aligned} 0 &= (v + w) \overline{\wedge} (v + w) \\ &= v \overline{\wedge} v + v \overline{\wedge} w + w \overline{\wedge} v + w \overline{\wedge} w \\ &= v \overline{\wedge} w + w \overline{\wedge} v. \end{aligned}$$

(2) Consider $v_1 \overline{\wedge} \cdots \overline{\wedge} v_k \in \Lambda^k(V)$. We can write each v_i as $v_i = c_i^j E_j$. We then have

$$\begin{aligned} v_1 \overline{\wedge} \cdots \overline{\wedge} v_k &= \left(\sum_{j=1}^{n} c_1^j E_j \right) \overline{\wedge} \cdots \overline{\wedge} \left(\sum_{j=1}^{n} c_k^j E_j \right) \\ &= \sum_{j_1, \cdots, j_k = 1}^{n} c_1^{j_1} \cdots c_k^{j_k} E_{j_1} \overline{\wedge} \cdots \overline{\wedge} E_{j_k}. \end{aligned}$$

Consider the term $E_{j_1} \overline{\wedge} \cdots \overline{\wedge} E_{j_k}$. Since $k > n$ we must have that $E_{j_l} = E_{j_k}$ for some $l \neq k$. Hence,

$$E_{j_1} \overline{\wedge} \cdots \overline{\wedge} E_{j_k} = 0$$

by (1). Since each summand is zero, we have that $v_1 \overline{\wedge} \cdots \overline{\wedge} v_k = 0$.

This completes the proof. □

Lemma 4.2 implies that we get an analogous direct sum decomposition

$$\overline{\Lambda}(V) = \bigoplus_{k=0}^{n} \overline{\Lambda}^k(V)$$

As discussed at the start of this section, the motivation behind the definition of the exterior algebra is to generalize the construction of the previous section. This is indeed the case as shown by Proposition 4.5. We first introduce a definition and a lemma.

Definition 4.3. Let V, W $\mathbb{R}$-vector spaces. A **pairing** between V and W is a bilinear map

$$\langle , \rangle : V \times W \to \mathbb{R}$$

A pairing is **perfect** if for every $v \in V$, there is a $w \in W$ such that $\langle v, w\rangle \neq 0$, and vice versa.

Lemma 4.4. *Let V and W be finite-dimensional $\mathbb{R}$-vector spaces The following statements are equivalent:*

(1) *There exists an isomorphism $W \cong V^*$,*
(2) *There exists a perfect pairing $\langle , \rangle : V \times W \to \mathbb{R}$.*

Proof. Assume that (1) is true. There is an obvious perfect pairing between V and V^*, given by

$$\langle , \rangle : V \times V^* \to \mathbb{R}, \quad \langle v, \varphi\rangle = \varphi(v)$$

Composing this with the obvious map $V \times W \to V \times V^*$, we see that (1) implies (2). Now assume that (2) is true. Given $w \in W$, we get a linear map $\varphi : V \to \mathbb{R}$ by sending v to $\varphi(v) = \langle v, w\rangle$. It is easy to see that this gives us a linear map $W \to V^*$, and since we have a perfect pairing, this map is injective. Since the vector spaces are finite-dimensional, we have that the linear map $W \to V^*$ is bijective, an isomorphism. Hence, (2) implies (1). □

We now prove the desired result.

Proposition 4.5. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. For each $0 \leq k \leq n$, we have*

$$\Lambda^k(V^*) \cong (\overline{\Lambda}^k(V))^* \cong \overline{\Lambda}^k(V^*)$$

Proof. (Sketch) Consider the following map:

$$\begin{aligned} \langle \cdot, \cdot\rangle : \overline{\Lambda}^k(V) \times \Lambda^k(V^*) \\ (v_1 \wedge \cdots \wedge v_k, \varepsilon^I) \mapsto \varepsilon^I(v_1 \wedge \cdots \wedge v_k) \end{aligned}$$

This is a valid map since any alternating k-multi-linear map in $\Lambda^k(V^*)$ is a linear combination of ε^I. It is clear that $\langle \cdot, \cdot\rangle$ is a pairing. Moreover, it is also a perfect pairing. The first isomorphism follows from Lemma 4.4. The second isomorphism is given by:

$$\varphi^1 \wedge \cdots \wedge \varphi^k \mapsto \big(v_1 \wedge \cdots \wedge v_k \mapsto \det\big(\varphi^i(v_j)\big)\big)\,.$$

It can be easily checked that this is an isomorphism. □

Remark 4.6. *Based on Proposition 4.5 we can now write $\overline{\Lambda}$ as Λ.*

4.2. **Complexification of Exterior Algebra.** We now study the complexififcation of $\Lambda(V)$. We first prove a basic lemma.

Lemma 4.7. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} = n$. Furthermore, assume that $V = W_1 \oplus W_2$, where W_1, W_2 are $\mathbb{R}$-vector subspaces of V such that $\dim_{\mathbb{R}} W_i = m_i$. Then*

(1) *We have*

$$(\Lambda^k V)^{\mathbb{C}} \cong \Lambda^k V^{\mathbb{C}}$$

(2) *We have*

$$\Lambda(W_1 \oplus W_2) \cong \Lambda W_1 \otimes_{\mathbb{R}} \Lambda W_2$$

(3) *For each* $0 \le k \le n$, *we have*

$$\Lambda^k(W_1 \oplus W_2) \cong \bigoplus_{p+q=k} (\Lambda^p W_1 \otimes_{\mathbb{R}} \Lambda^q W_2)$$

Proof. The proof is given below:

(1) First fix some $z \in \mathbb{C}$ and consider the the map

$$\underbrace{V \times \cdots \times V}_{k-\text{times}} \to \Lambda^k V^{\mathbb{C}}$$
$$(v_1, \dots, v_k) \mapsto z \cdot (v_1 \otimes 1) \wedge \cdots \wedge (v_k \otimes 1).$$

This map is an alternating k-multi-linear map. Hence it descends to an $\mathbb{R}$-linear map

$$\phi_z : \Lambda^k V \to \Lambda^k(V^{\mathbb{C}})$$
$$v_1 \wedge \cdots \wedge v_k \mapsto z \cdot (v_1 \otimes 1) \wedge \cdots \wedge (v_k \otimes 1).$$

Here we have used the universal property of the tensor product and the k-th exterior power. Now, we define an $\mathbb{R}$-bilinear map

$$T : \Lambda^k V \times \mathbb{C} \to \Lambda^k(V^{\mathbb{C}})$$
$$(\omega, z) \mapsto \phi_z(\omega).$$

Since T is $\mathbb{R}$-bilinear, it lifts to an $\mathbb{R}$-linear map

$$\overline{T} : \Lambda^k V \otimes_{\mathbb{R}} \mathbb{C} \to (\Lambda^k V)^{\mathbb{C}} \to \Lambda^k(V^{\mathbb{C}})$$
$$(\omega \otimes z) \mapsto \phi_z(\omega).$$

The map $\overline{T}$ is defined as

$$\overline{T}\big((v_1 \wedge \ldots \wedge v_k) \otimes z\big) = z \cdot (v_1 \otimes 1) \wedge \ldots \wedge (v_k \otimes 1)$$

It is easy to that T is $\mathbb{C}$-linear isomorphism.

(2) (Sketch) We take for granted the statement that $\Lambda(-)$ is a functor from the category of $\mathbb{R}$-vector spaces to the category of graded-commutative $\mathbb{R}$-algebras, such that $\Lambda(-)$ is left adjoint to the functor that takes the degree 1 part. Since $\Lambda(-)$ is is left adjoint, it preserves colimits, in particular direct sums. Hence,

$$\Lambda(W_1 \oplus W_2) \cong \Lambda W_1 \otimes_{\mathbb{R}} \Lambda W_2$$

(3) This follows from (2). Indeed, (2) implies

$$\begin{aligned}\bigoplus_{k=0}^{n} \Lambda^k(W_1 \oplus W_2) &= \Lambda(W_1 \oplus W_2) \\ &= \Lambda W_1 \otimes_{\mathbb{R}} \Lambda W_2 \\ &= \left(\bigoplus_{p=0}^{m_1} \Lambda^p W_1\right) \otimes_{\mathbb{R}} \left(\bigoplus_{q=0}^{m_2} \Lambda^q W_2\right) = \bigoplus_{p,q=0}^{m_1,m_2} \Lambda^p W_1 \otimes_{\mathbb{R}} \Lambda^q\end{aligned}$$

Recall the Vandermode identity

$$\binom{n}{k} = \binom{m_1 + m_2}{k} = \sum_{l=0}^{k} \binom{m_1}{l}\binom{m_2}{k-l} = \sum_{p+q=k} \binom{m_1}{p}\binom{m_2}{q} W_2$$

If we identify the degree k-component of each side of the equation with a vector subspace of dimension $\binom{n}{k}$, we have, by comparing the degree k-component, that

$$\Lambda^k(W_1 \oplus W_2) = \bigoplus_{p+q=k} \Lambda^p W_1 \otimes_{\mathbb{R}} \Lambda^q W_2$$

This completes the proof □

Corollary 4.8. *Let V be a $\mathbb{R}$-vector space such that* $\dim_{\mathbb{R}} = n$. *Then*

$$(\Lambda^k V)^{\mathbb{C}} = \bigoplus_{p+q=k} (\Lambda^p V_{(1,0)} \otimes_{\mathbb{C}} \Lambda^q V_{(0,1)})$$

Proof. Using Lemma 4.7, we can do the following computation:

$$\begin{aligned}(\Lambda^k V)^{\mathbb{C}} &= \Lambda^k V^{\mathbb{C}} \\ &= \Lambda^k (V_{(1,0)} \oplus V_{(0,1)}) \\ &= \bigoplus_{p+q=k} \left(\Lambda^p V_{(1,0)} \otimes_{\mathbb{C}} \Lambda^q V_{(0,1)}\right).\end{aligned}$$

This completes the proof. □

We will use Lemma 4.7 as a starting point for the study of differential forms on a complex manifold. Additional details regarding the complexification of the exterior algebra, such as the Hodge inner product, will be introduced later on.

Part 2. Sheaves

A key tool in smooth manifold theory is the existence of partitions of unity. Partitions of unity allow us to glue together local constructions to form global constructions on a smooth manifold. For example, a partition of unity can be used to prove the existence of a Riemannian metric on a smooth manifold. Recall that smooth functions used to construct a partition of unity have compact support. By the Identity Theorem (Proposition 1.10), any holomorphic function on an open subset of $\mathbb{C}^n$ with compact support must be identically zero. Hence, we cannot have holomorphic partitions of unity on open subsets of $\mathbb{C}^n$ (and more generally, on complex manifolds). This problem motivates the study of sheaves of holomorphic functions. Their applications to complex manifolds are numerous.

5. Pre-Sheafs & Sheafs

A topological space, X, can be studied by studying the algebra of continuous functions on X. However, a generic topological space, such as a non-normal topological space, can have few globally defined functions. A more precise perspective then is that a topological space can be studied by by studying *locally defined* functions. This richer perspective is formalized using a mathematical object called a pre-sheaf.

Definition 5.1. Let X be a topological space. Let **Ab** be the category of abelian groups and let **Open**(X) be the category of open sets on X. A **pre-sheaf** on X with values on $\mathscr{C}$ is a contravariant functor:

$$\mathscr{F} : \mathbf{Open}(X) \to \mathbf{Ab}$$

What? Let's de-construct Definition 5.1. Let $X = \mathbb{C}^n$. Assign to each open set $U \subseteq \mathbb{C}^n$ the abelian group[3] of holomorphic functions on U, $\mathscr{F}(U)$. We also have a collection of restriction maps $r^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ whenever $U \supseteq V$. These maps satisfy the following properties:

(1) $r^U_U = \mathrm{Id}_U$ for every open set U,
(2) $r^V_W \circ r^U_V = r^U_W$ whenever $U \supseteq V \supseteq W$.

The properties of the restriction map is exactly encoded by the requirement that $\mathscr{F}$ is a contravariant functor from $\mathbf{Open}(X)$ to $\mathbf{Ab}$.

Remark 5.2. *If $\mathscr{F}$ is a pre-sheaf on X, we refer to $\mathscr{F}(U)$ as the sections of the pre-sheaf $\mathscr{F}$ over the open set U. We sometimes use the notation $\Gamma(U, \mathscr{F})$ to denote $\mathscr{F}(U)$. If $V \subseteq U$, we write ρ^U_V, r^U_V, res^U_V or $|_V$ for the morphism between $\mathscr{F}(V)$ and $\mathscr{F}(U)$.*

We can replace $\mathbf{Ab}$ with the category of commutative rings, $\mathbf{CRing}$. This leads to the following definition.

Definition 5.3. Let X be a topological space and let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups and commutative rings respectively. $\mathscr{F}$ is a **sheaf of $\mathscr{G}_X$-modules** (or a $\mathscr{G}_X$-module sheaf) if for each open set $U \subseteq X$, $\mathscr{F}(U)$ is an $\mathscr{G}_X(U)$-module, such that the restriction maps on $\mathscr{F}$ are compatible with the module structures induced by the restriction maps in $\mathscr{G}_X$.

Remark 5.4. *We will primarily focus on the category of abelian groups or R-modules. Since the discussion is similar in both cases, we will concentrate on the category of abelian groups for the most part.*

Given a pre-sheaf on X, a natural question to ask is the extent to which its sections over an open set $U \subseteq X$ are determined by their restrictions to the open subsets of U. A sheaf is roughly speaking a pre-sheaf where the aforementioned question can be answered affirmatively.

Definition 5.5. Let X be a topological space. A **sheaf** of abelian groups on X is a pre-sheaf of abelian groups that satisfies the following two conditions:

(1) (**Identity Axiom**) If $\{U_i\}_{i\in I}$ is an open cover of U, and $f_1, f_2 \in \mathscr{F}(U)$, and $f_1|_{U_i} = f_2|_{U_i}$ for all i, then $f_1 = f_2$.
(2) (**Gluing Axiom)** Suppose $\{U_i\}_{i\in I}$ is an open cover of U. Suppose for each i we have $f_i \in \mathscr{F}(U_i)$ such that $f_i = f_j$ in $\mathscr{F}(U_i \cap U_j)$. Then there is a unique $s \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$.

Example 5.6. The following is a list of some examples of sheaves:

(1) If X is a topological space, the pre-sheaf of continuous functions, $\mathscr{C}$, defined by $U \mapsto \mathscr{C}(U)$, where $\mathscr{C}(U)$ is the abelian group of continuous functions on U (with usual restrictions), is a sheaf.
(2) If X is a topological space, the pre-sheaf of nowhere vanishing continuous functions, $\mathscr{C}^\times$, defined by $U \mapsto \mathscr{C}^\times(U)$, where $\mathscr{C}^\times(U)$ is the abelian group of no-where vanishing continuous functions on U (with usual restrictions), is a sheaf.
(3) If $X = \mathbb{C}^n$, the pre-sheaf of holomorphic functions, $\mathscr{O}$, defined by $U \mapsto \mathscr{O}(U)$, where $\mathscr{O}(U)$ is the abelian group of holomorphic functions on U (with usual restrictions), is a sheaf.

[3]In fact, this is a commutative ring.

(4) If $X = \mathbb{C}^n$, the pre-sheaf of nowhere vanishing holomorphic functions, $\mathscr{O}^\times$, defined by $U \mapsto \mathscr{O}^\times(U)$, where $\mathscr{O}^\times(U)$ is the abelian group of non-where vanishing holomorphic functions on U (with usual restrictions), is a sheaf.

Remark 5.7. *We can easily generalize Example 5.6 by considering the sheaf of functions restricted to open subsets of the appropriate space.*

Example 5.8. Let X be a topological space and let A be an abelian group with the discrete topology. The following are two examples of pre-sheafs of abelian groups:

(1) For any open set $U \in \mathbf{Open}(X)$, let $\overline{A}(U) = A$. Clearly, $\overline{A}$ is a pre-sheaf with restriction maps the identity. This is called the **constant pre-sheaf**.
(2) For any open set $U \in \mathbf{Open}(X)$, let $\underline{A}(U)$ be the abelian group of all continuous maps of U into A. Then with the usual restriction maps (as in the previous example), we obtain a sheaf. Note that each function in $\underline{A}(U)$ is locally constant for each open set of X. This is called the **constant sheaf**.

Remark 5.9. *Let $\underline{A}$ be the constant sheaf. Note that for every connected open set U, $\underline{A}(U) \cong A$ since the image of a continuous map from a connected set to a discrete space is constant. This justifies the terminology.*

Remark 5.10. *Let X be a topological space. Let $\mathscr{G}$ be a sheaf of commutative rings on X and let $\mathscr{F}$ be a sheaf of $\mathscr{G}_X$-modules. We will write that $\mathscr{F}$ is a sheaf of R-modules if $\mathscr{G}$ is the constant pre-sheaf $\overline{R}$ of commutative rings. Several statements below are stated specifically for a sheaf of R-modules, but more generally, they apply to sheaves of $\mathscr{G}_X$-modules.*

Example 5.11. All examples discussed in Example 5.6 are examples of sheaves of R-modules with $R = \mathbb{R}, \mathbb{C}$ as appropriate.

Example 5.12. (Sheaf of Sections) Let X, Y be topological space and let $\pi : Y \to X$ be a continuous map. Recall that a section of π is a continuous map $\sigma : X \to Y$ such that $\pi \circ \sigma = \mathrm{Id}_X$. For an open non-empty set $U \subseteq X$, define $\mathscr{E}(U)$ to be the set of sections of π on U. That is,

$$\mathscr{E}(U) = \{\sigma : U \to Y \mid \sigma \text{ is continuous}, \pi \circ \sigma = \mathrm{Id}_U\}.$$

The empty set is sent to the singleton set and the restriction maps are are given by restriction of functions. This is called the pre-sheaf of sections of π. In fact, pre-sheaf of sections of π is a sheaf of sets. Indeed, since sections are indeed continuous function, it is clear that the identity axiom is satisfied. Similarly, the gluing axiom is also satisfied if we note that if $\{U_i\}_{i\in I}$ is an open cover of of U and $\sigma_i \in \mathscr{E}(U_i)$ such that $\sigma_i = \sigma_j \in \mathscr{E}(U_i \cap U_j)$, then the function $\sigma : U \to Y$ such that $\sigma|_{U_i} = \sigma_i$ is indeed a section.

Example 5.13. We can also define sheaves which don't quite look like sheaves of functions. Let X be a topological space and let $*$ denote the trivial abelian group. Fix any abelian group, A, and $x \in X$. Consider the assignment

$$i_x^A(U) = \begin{cases} A & \text{if } x \in U, \\ * & \text{if } x \notin U. \end{cases}$$

This is can be made into a pre-sheaf if for open sets $U, V \subseteq X$ such that $V \subseteq U$, the map $i_x^A(U) \to i_x^A(V)$ is defined such that:

(1) If $x \notin U$, then the map is simply the identity morphism $* \to *$
(2) If $x \in V$, then the map is simply the identity morphism $A \to A$

(3) If $x \in U \setminus V$, then the map is simply the unique morphism $A \to *$.

It is easy to verify that this defines a pre-sheaf. Let $\{U_i\}_{i \in I}$ for an open cover for an open set $U \subseteq X$. The identity and gluing axioms are essentially satisfied since U contains x if and only if some U_i contains x. This is called the **skyscraper sheaf**.

Remark 5.14. *A pre-sheaf may not be a sheaf. Let $X = \mathbb{R}$, and let $\mathscr{F}(U)$ be the set of all bounded functions on U. Then $\mathscr{F}$ defines a pre-sheaf but not a sheaf. Indeed, let $X = \bigcup_{i \in \mathbb{Z}}^{\infty}(i, i+1]$ and let $s_i \equiv i$ on $(i, i+1]$. Since $V_i \cap V_j = \emptyset$ for $i \neq j$, trivially we have that $s_i = s_j$ on each $V_i \cap V_j = \emptyset$ for $i \neq j$. However, there is not $s \in \mathscr{F}(\mathbb{R})$ such that $s|_{(i,i+1]} = s_i$; otherwise, s must be an unbounded function.*

Category theory teaches us to always define morphisms between mathematical objects. We now define morphisms of presheaves, and similarly for sheaves. In other words, we will describe the category of presheaves and the category of sheaves

Definition 5.15. Let X be a topological space and let $\mathscr{F}$ and $\mathscr{G}$ be pre-sheafs of abelian groups. A **morphism of pre-sheafs**, $\varphi : \mathscr{F} \to \mathscr{G}$, is a natural transformation.

What? Recalling the definition of a natural transformation, Definition 5.15 means that for each open set $U \subseteq X$ there exists a morphism from $\mathscr{F}(U) \to \mathscr{G}(U)$ such that whenever $V \subseteq U$, the following diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}_V^U"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_V^U"] \\ \mathscr{F}(V) \arrow[r, "\varphi(V)"'] & \mathscr{G}(V) \end{tikzcd}$$

commutes. Definition 5.15 makes the collection all pre-sheaves on X into a category, which we denote as $\mathbf{PreShv}(X, \mathbf{Ab})$. The category of sheaves on X, which we denote as $\mathbf{Shv}(X, \mathbf{Ab})$, is then a full subcategory of the category of presheaves on X satisfying the identity and gluing axioms.

Example 5.16. The following is a list of examples of morphisms of pre-sheafs and sheafs.

(1) Let X be a topological space and G and H be abelian groups. Let $\mathscr{G}$ and $\mathscr{H}$ be the corresponding constant sheaves. Every group homomorphism $F : G \to H$ defines a sheaf morphism $\mathcal{F} \colon \mathcal{G} \to \mathcal{H}$ given by $\mathcal{F}(U)(f) = F \circ f$.

(2) Let $X = \mathbb{C}^n$. If U is an open subset of $\mathbb{C}^n$, there is a map of abelian groups

$$\epsilon : \mathscr{O}(U) \to \mathscr{O}^*(U), \quad \epsilon(U)(f) = e^{2\pi i f}$$

This defines a sheaf morphism $\varepsilon : \mathscr{O} \to \mathscr{O}^*$

6. Stalks

The stalk of a pre-sheaf captures local data of a pre-sheaf. Let's consider a concrete example.

Example 6.1. Let $X = \mathbb{C}^n$ and let $\mathscr{O}$ denotes the sheaf of holomorphic functions on X. For each $x \in X$ and open set U containing x, we define an equivalence relation on $\mathscr{O}(U)$

$$f \sim g \iff \text{there exists an open set } W \subseteq U \text{ containing } p \text{ such that } f|_W = g|_W$$

The equivalence class of a function $f \in \mathscr{O}(U)$ is called the germ of f at x and is denoted by $[f]_x$. The stalk of $\mathscr{O}$ at x, denoted $\mathscr{O}_x$, is the vector space of all germs of holomorphic functions at x. Addition and scalar multiplication of germs are defined by performing these operations on any representatives that are defined on the same open set. For example, addition is defined as:

$$[f]_x + [g]_x = [f+g]_x$$

Let's check that addition is well-defined. Assume that $[f]_x = [f']_x$ and $[g]_x = [g']_x$. Then there exist open sets $V, W \subseteq U$ such that $f|_V = f'|_V$ and $g|_W = g'|_W$. It is clear that on $V \cap W \subseteq U$, we have

$$f + g|_{V\cap W} = f' + g'|_{V\cap W}$$

This shows addition is well-defined. Similarly, it can be checked that scalar multiplication is well-defined.

Remark 6.2. *$\mathscr{O}_x$ is actually a ring. This can be checked easily. In fact, $\mathscr{O}_x$ is a local ring. Let $\mathfrak{m}_x \subseteq \mathscr{F}_x$ denotes germs vanishing at p. This certainly forms an ideal. In fact, the ideal is maximal since $\mathscr{F}_x/\mathfrak{m}_x \cong \mathbb{C}$. This is the unique maximal ideal since any germ not contained in $\mathfrak{m}_x$ is invertible.*

Clearly, this construction can similarly be applied to continuous or smooth functions on an appropriate space. We can now give the general definition of a stalk of a pre-sheaf, abstracting away from the previous example.

Definition 6.3. Let X be a topological space and let $\mathscr{F}$ be a pre-sheaf on X. The **stalk** of $\mathscr{F}$ at $x \in X$, denoted by $\mathscr{F}_x$, is the direct limit

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$$

Remark 6.4. *The stalk of a sheaf is the stalk of the underlying pre-sheaf.*

What? Let's understand the finer details of the definition. Recall that a directed set $(I, \le)$ is a non-empty set I with a binary relation, $\le$, that is reflexive and transitive, and where every pair of elements has a common upper bound. A direct system of abelian groups consists of a family $\{G_\alpha\}_{\alpha\in I}$ of objects indexed by a directed set I, along with homomorphisms $f_{\alpha\beta} : G_\alpha \to G_\beta$ for $\alpha \le \beta$, satisfying

$$f_{\alpha\alpha} = \mathrm{Id}_{G_\alpha}, \alpha \in I \qquad f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma},\ \alpha \le \beta \le \gamma$$

The direct limit (or colimit in this case) is defined by defining an equivalence relation on $\coprod_{\alpha\in I} G_\alpha$ such that

$$g_\alpha \sim g_\beta \iff \text{there exists some } \gamma \in I \text{ such that } \alpha, \beta \le \gamma \text{ and } f_{\alpha\gamma}(g_\alpha) = f_{\beta\gamma}(g_\beta) \in G_\gamma$$

The direct limit of the direct system is denoted as

$$\varinjlim_{\alpha\in I} G_\alpha = \left(\coprod_{\alpha \in I} G_\alpha \right) / \sim$$

$\varinjlim_{\alpha\in I} G_\alpha$ is an abelian group with addition defined by

$$[g_\alpha] + [g_\beta] = [f_{\alpha\gamma}(g_\alpha) + f_{\beta\gamma}(g_\beta)],$$

where γ is some upper bound for α and β. This can be checked to be well-defined because all maps $f_{\alpha\beta}$ are homomorphisms. We can now make sense of Definition 6.3. If we work in **Ab**, the characterization of co-limit of a directed system allows us to unpack the definition of the stalk of a pre-sheaf. Let $\mathscr{F}$ be a **Ab**-valued pre-sheaf on a topological space X. For each $x \in X$, the collection of abelian groups $\mathscr{F}(U)$, where U ranges over all open sets containing x, together with the restriction maps, forms a direct system with the relation $U \leq V$ if $U \supseteq V$. The intersection of two open sets containing p serves as a common upper bound. Definition 6.3 defines the stalk of $\mathscr{F}$ at p as the direct limit of this system.

Category theory teaches us to focus on the properties of morphisms between objects rather than the objects themselves. Consequently, we define the concept of the stalk of a morphism of sheaves.

Definition 6.5. Let X be a topological space and let $\mathscr{F}, \mathscr{G}$ be sheaves on X. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism, and let $\mathscr{F}_x$ be a stalk of $\mathscr{F}$ at $p \in X$. The **stalk of the morphism** at $p \in X$ is the morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ such that $\varphi_x([f]_x) = [\varphi(f)]_x$.

Let's check that Definition 6.5 is well-defined. Suppose $[f]_x = [f']_x$ such that $f \in \mathscr{F}(U)$ and $f' \in \mathscr{F}(U')$. Then there exists an open set $W \subseteq U \cap U'$ containing p such that $f|_W = f'|_W$. We have

$$\mathscr{F}(f)|_W = \mathscr{F}(f|_W) = \mathscr{F}(f'|_W) = \mathscr{F}(f')|_W$$

Hence $[\mathscr{F}(f)]_x = [\mathscr{F}(f')]_x$. It is easy to check that $\mathscr{F}_x$ is a homomorphism of abelian groups.

7. Étalé Space & Sheafification

7.1. **Étalé Space.** Using the concept of stalks, we define a topological space naturally associated with each pre-sheaf, referred to as the Étalé space of the pre-sheaf. An Étalé space over a topological space X is a topological space E together with a local homeomorphism

$$\pi \colon E \to M$$

The preimage $E_x = \pi^{-1}(x)$ of a point $x \in M$ is called the stalk of E over x. An Étalé space is called an Étalé space of abelian groups if each stalk has an abelian group structure and the operations are continuous. We now see that we can associate an Étalé space associated to every pre-sheaf.

Proposition 7.1. *Let X be a topological space and let $\mathscr{F}$ be a pre-sheaf of abelian groups over X. Let $\mathrm{Et}(\mathscr{F})$ be the disjoint union of the stalks $\mathscr{F}_x$ for all $p \in X$, with the projection $\pi \colon \mathrm{Et}(\mathscr{F}) \to X$ defined by $\pi([f]_x) = x$. For each open set $U \subseteq X$ and each $f \in \mathscr{F}(U)$, define a map $f^+ \colon U \to \mathrm{Et}(\mathscr{F})$ by*

$$f^+(x) = [f]_x.$$

$\mathrm{Et}(\mathscr{F})$ has a unique topology such that π is a local homeomorphism and each f^+ is continuous section of π.

Remark 7.2. *Let $\pi : E \to X$ be a local homeomorphism. Recall the following facts:*

- *For any open set $U \subseteq X$ and any section $s : U \to E$, the image $s(U)$ is open in E, and homeomorphic to U via s. That is,*
$$s = (\pi|_{s(U)})^{-1}.$$
- *Sets of the form $s(U)$, where U ranges over open subsets of X and s ranges over sections of U, form a basis of the topology on E.*

Philosophically, the topology on E is determined by the topology on X and the sections of π. This motivation is the starting point of the proof of Proposition 7.1.

Proof. We use the collection of all subsets of the form

$$f^+(U) = \{[f]_x : x \in U\}$$

where $U \subseteq X$ is an open set and $f \in \mathscr{F}(U)$ as a basis for a topology on $\mathrm{Et}(\mathscr{F})$. We need to check two conditions:

(1) Every point of $\mathrm{Et}(\mathscr{F})$ is contained in some $f^+(U)$.
(2) If two basis sets $f^+(U)$ and $g^+(V)$ intersect at a point $[g]_x$, then there exists a basis set $h^+(W)$ such that $[h]_x \in h^+(W) \subseteq f^+(U) \cap g^+(V)$.

Condition (1) is straightforward: every germ $[f]_x$ is represented by some section $f \in \mathscr{F}(U)$, and hence $[f]_x$ is an element of the basis set $f^+(U)$. For condition (2), suppose $[h]_x \in f^+(U) \cap g^+(V)$; this implies $p \in U \cap V$ and that there exists a neighborhood $W \subseteq U \cap V$ of p such that $h|_W = f|_W = g|_W$, and thus $[h]_x \in w^+(W) \subseteq f^+(U) \cap g^+(V)$ as required. Proof of the uniqueness of the topology is skipped.

To verify that each map $f^+ : U \to \mathrm{Et}(\mathscr{F})$ is continuous, let $g^+(V) \subseteq \mathrm{Et}(\mathscr{F})$ be a basis open set and observe that

$$\begin{aligned}(f^+)^{-1}(g^+(V)) &= \{x \in U \cap V : [f]_x = [g]_x\} \\ &= \{x \in U \cap V : f|_W = g|_W \text{ for some neighborhood } W \text{ of } x\},\end{aligned}$$

which is open in U. To show π is a local homeomorphism it suffices to show that the restriction of π to each basis open set $f^+(U)$ is a homeomorphism onto U. To see that it is continuous, let $V \subseteq U$ be open. Then $\pi^{-1}(V) \cap f^+(U)$ is the set of germs of f at points of V, which is exactly the basis set $(f|_V)^+(V)$. On the other hand, $\pi|_{f^+(U)} : f^+(U) \to U$ has a continuous inverse given by the local section $f^+ : U \to f^+(U)$, so it is a homeomorphism onto its image.

If $\mathscr{F}$ is a pre-sheaf of abelian groups, then each stalk $\mathscr{F}_x$ inherits the structure of an abelian group via the direct limit construction as discussed above. For example, let

$$a : \mathrm{Et}(\mathscr{F}) \times_X \mathrm{Et}(\mathscr{F}) \to \mathrm{Et}(\mathscr{F})$$

be addition. We check addition is continuous. Suppose $f^+(U) \subseteq \mathrm{Et}(\mathscr{F})$ is a basis open set. Then $a^{-1}(f^+(U))$ is the set of all pairs of the form $([g]_x, [h]_x)$ where $x \in U$, $g, h \in \mathscr{F}(W)$ for some neighborhood $x \in W \subseteq U$ and $g + h = f|_W$. That is to say,

$$a^{-1}(f^+(U)) = \bigcup_{x \in W \subseteq U} \bigcup_{\substack{g, g \in \mathscr{F}(V) \\ g+h = f|_W}} g^+(W) \times h^+(W).$$

As a union of open sets, this is open. Similar arguments show that the other appropriate algebraic operations are continuous. □

Let X be a fixed topological space. For any topological space Y, a continuous function $f : Y \to X$ is called a space over X, or a bundle over X. The category of bundles over X is defined as the slice category $\mathbf{Top}/X$, where X is called the base space. We have already seen that a *bundle-like* structure gives rise to a sheaf of sections. Moreover, the Étalé space construction emphasizes that a pre-sheaf gives rise to a *bundle-like* structure. Let's formalize in the language of functors. Define

$$\begin{aligned}\Gamma : \mathbf{Top}/X &\to \mathbf{PreShv}(X, \mathbf{Ab}) \\ Y &\mapsto \mathscr{E}_Y\end{aligned}$$

that associates to $Y \in \mathbf{Top}/X$ the sheaf of sections. We have already seen that Γ is well-defined. Let's verify that $\mathscr{E}$ is a functor. Let $Y, Y' \in \mathbf{Top}/X$ and suppose we have a morphism $p : Y \to Y'$ in $\mathbf{Top}_X$:

$$\begin{tikzcd}
Y \arrow[rr, "p"] \arrow[dr, "f"'] & & Y \arrow[dl, "f'"] \\
& X &
\end{tikzcd}$$

We want to show that it gives rise to a sheaf morphism $\mathscr{E}_Y \to \mathscr{E}_{Y'}$. Indeed, given any open set $U \subseteq X$, we have a morphism $\mathscr{E}_Y(U) \to \mathscr{E}_{Y'}(U)$ defined by the following diagram.

$$\begin{tikzcd}
& Y \arrow[r, "p"] \arrow[d, "f"] & Y' \arrow[d, "f'"'] \\
U \arrow[ur, "\sigma"] \arrow[r, hook] & X \arrow[r, "\mathrm{Id}_X"] & X
\end{tikzcd}$$

The naturality condition is obvious. Hence, $\mathscr{E}_Y \to \mathscr{E}_{Y'}$ is a sheaf morphism and Γ is a functor. The Étalé space allows us to define

$$\Lambda : \mathbf{PreShv}(X, \mathbf{Ab}) \to \mathbf{Top}/X$$
$$\mathscr{F} \mapsto \mathrm{Et}(\mathscr{F})$$

Let $\mathscr{F}, \mathscr{G} \in \mathbf{PreShv}(X, \mathbf{Ab})$ and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a pre-sheaf morphism. For each $x \in X$, the stalkification functor defines a morphism $\mathscr{F}_x \to \mathscr{G}_x$ of stalks. This allows us to define the obvious map $\Lambda(\varphi) : \mathrm{Et}(\mathscr{F}) \to \mathrm{Et}(\mathscr{G})$ by defining the stalkification map pointwise on the disjoint union. We check that $\Lambda(\varphi)$ is a continuous function. Consider $f^+(U) \subseteq \mathrm{Et}(\mathscr{F})$. Note that we have

$$g_x \in (\Lambda(\varphi))^{-1}(f^+(U)) \iff \varphi_x(t_x) \in f^+(U) \iff \varphi_x(g_x) = f_x.$$

Thus,

$$(\Lambda(\varphi))^{-1}(f^+(U)) = \bigcup_{V \subseteq U} \bigcup_{g \in \mathscr{F}(V)} t^+(\{x \in V \mid (\varphi(V)(g))_x = f_x\}).$$

Since the sets on the right hand side are open, $(\Lambda(\varphi))^{-1}(f^+(U))$ is an open set. Hence, $\Lambda(\varphi)$ is a continuous function. This shows that Λ is a functor. We have the following proposition:

Proposition 7.3. *The functor Λ is left-adjoint to the functor Γ. That is, there is a bijection*

$$\mathrm{Hom}_{\mathbf{Shv}}(\Lambda(\mathscr{F}), \mathscr{G}) \cong \mathrm{Hom}_{\mathbf{Top}/X}(Y, \Gamma(Y'))$$

Proof. □

We can define a subcategory of $\mathbf{Et}(X) \subseteq \mathbf{Top}/X$ of Étalé spaces that includes all objects in $\mathbf{Top}/X$ but only includes local homeomorphisms between objects in $\mathbf{Top}/X$ as morphisms. We have the following result:

Proposition 7.4. *Let X be a topological space. The categories $\mathbf{Shv}(X, \mathbf{Ab})$ and $\mathbf{Et}(X)$ are equivalent.*

Proof. □

7.2. **Sheafification.** Recall that a pre-sheaf may not be a sheaf. Indeed, Remark 5.14 furnishes an example. This motivates the definition of sheafification. Note that the Étalé space construction plays an important role in the definition of a sheafification functor.

Proposition 7.5. *Let X be a topological space and $\mathscr{F}$ is a pre-sheaf on X of abelian groups. There is a sheaf $\mathscr{F}^+$ of abelian groups on X, called the **sheafification** of $\mathscr{F}$, together with a canonical pre-sheaf morphism $\theta_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^+$ satisfying the following properties:*

1. *If $\mathscr{G}$ is a sheaf on X and $\varphi : \mathscr{F} \to \mathscr{G}$ is a pre-sheaf morphism, then there exists a unique sheaf morphism $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$ such that the following diagram commutes:*

$$\begin{tikzcd} & \mathscr{F} \arrow[dl, "\theta_{\mathscr{F}}"'] \arrow[d, "\varphi"] \\ \mathscr{F}^+ \arrow[r, dashed, "\varphi^+"'] & \mathscr{G} \end{tikzcd}$$

2. *$\theta_{\mathscr{F}}$ is an isomorphism if and only if $\mathscr{F}$ is a sheaf.*

Remark 7.6. *For intuition, let $\mathscr{F}$ be a pre-sheaf of functions. Recall that each $f \in \mathscr{F}(U)$ is determined by its behavior on the stalks $\mathscr{F}_x$ for $x \in X$. If $\mathscr{F}$ is not a sheaf, we would like to restrict to those $f \in \mathscr{F}(U)$ such that the gluing axiom holds. A possible candidate for the definition of $\mathscr{F}^+(U)$ for each open set $U \subseteq X$ is:*

$$\mathscr{F}^+(U) = \left\{ f = (f_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x \;\middle|\; \forall x \in U, \exists V_x \subseteq U \text{ and } g \in \mathscr{F}(V_x) \text{ s.t. } f_y = g_y \ \forall y \in V_x \right\}$$

A moment's reflection by using the definition of the topology on the Étalé space. shows that this can be done by taking continuous sections of the Étalé space.

Proof. Define the sheaf $\mathscr{F}^+$ on X by letting $\mathscr{F}^+(U)$ be the set of local sections of $\mathrm{Et}(\mathscr{F})$ over the open set $U \subseteq X$. The algebraic operations on $\mathscr{F}^+$ are defined pointwise: for example, we define a group structure on each set $\mathscr{F}^+(U)$ by

$$(s_1 + s_2)(p) = s_1(p) + s_2(p).$$

If $U = \emptyset$, we just interpret $\mathscr{F}^+(\emptyset) = \{\emptyset\}$ to be the trivial group. We now prove the said properties:

1. We define the sheaf morphism $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}$ first by defining $\varphi^+(s^+) = \varphi(s) \in \mathscr{G}(U)$ for any open set $U \subseteq X$ and $s \in \mathscr{F}(U)$. This ensures that $\varphi^+ \circ \theta_{\mathscr{F}} = \varphi$. Then we extend φ^+ to act on an arbitrary section $f \in \mathscr{F}^+(U)$ as follows: given $x \in U$, the definition of $\mathrm{Et}(\mathscr{F})$ shows that $\varphi(x) = [s]_x$ for some section $s \in \mathscr{F}(W)$ on some neighborhood $x \in W$. Then $s^+(W)$ is a neighborhood of $[s]_p$ in $\mathrm{Et}(\mathscr{F})$, and since f is continuous, $V = f^{-1}(s^+(W))$ is a neighborhood of $x \in U$. This means $f(q) = [s]_q$ for all $q \in V \cap U$. Hence, there is an open cover $\{U_\alpha\}$ of U and sections $s_\alpha \in \mathscr{F}(U_\alpha)$ such that $f|_{U_\alpha} = s_\alpha^+$ for each α. Let $\varphi^+(f)$ be the element $\tau \in \mathscr{G}(U)$ such that $\tau|_{U_\alpha} = \varphi(s_\alpha)$ for all α. Any other such map F' would have to agree with this one.
2. This is clear.

This completes the proof. $\square$

We can now define the sheafification functor,

$$\mathrm{Sh} : \mathbf{PreShv}(X, \mathbf{Ab}) \to \mathbf{Shv}(X, \mathbf{Ab})$$

that assigns to each pre-sheaf, $\mathscr{F}$, on X a its sheafification, $\mathscr{F}^+$. Let's verify that sheafification is indeed a functor. If $\mathscr{G}$ is a pre-sheaf such that $f : \mathscr{F} \to \mathscr{G}$ is a pre-sheaf morphism, then the universal property in the definition implies that there is a unique morphism $\mathrm{Sh}(f)$ such that the diagram

$$\begin{tikzcd}
\mathscr{F}^+ \arrow[d, dashed, "\mathrm{Sh}(f)"'] & \mathscr{F} \arrow[l, "\mathrm{Sh}"'] \arrow[d, "f"] \arrow[dl, "\mathrm{Sh}\circ f"] \\
\mathscr{G}^+ & \mathscr{G} \arrow[l, "\mathrm{Sh}"]
\end{tikzcd}$$

commutes. It is clear that $\mathrm{Sh}(\mathrm{Id}_{\mathscr{F}}) = \mathrm{Id}_{\mathrm{Sh}(\mathscr{F})} = \mathrm{Id}_{\mathrm{Sh}(\mathscr{F}^+)}$. Moreover, if $\mathscr{G}$ and $\mathscr{H}$ are sheaves such that $f : \mathscr{F} \to \mathscr{G}$ and $g : \mathscr{G} \to \mathscr{H}$ are pre-sheaf morphisms, then $\mathrm{Sh}(g \circ f) = \mathrm{Sh}(g) \circ \mathrm{Sh}(f)$ since the following diagram commutes.

$$\begin{tikzcd}
\mathscr{F}^+ \arrow[d, "\mathrm{Sh}(f)"] \arrow[dd, bend right, "\mathrm{Sh}(f)\circ\mathrm{Sh}(f)"'] & \mathscr{F} \arrow[l, "\mathrm{Sh}"'] \arrow[d, "f"] \arrow[dd, bend left, "g\circ f"] \\
\mathscr{G}^+ \arrow[d, "\mathrm{Sh}(g)"] & \mathscr{F} \arrow[l, "\mathrm{Sh}"] \arrow[d, "g"] \\
\mathscr{H}^+ & \mathscr{H} \arrow[l, "\mathrm{Sh}"]
\end{tikzcd}$$

8. Operations on Sheaves

Let X be a topological space. We can make various *abelian group like* operations on pre-sheaves of abelian groups on X. The purpose of this section is to catalog a list of such operations on pre-sheaves. We start with a very basic example.

Example 8.1. (Restriction of a Pre-Sheaf) Let X be a topological space and suppose $\mathscr{F}$ is a pre-sheaf of abelian groups on X. Let $V \subseteq X$ be an open subset. The **restriction** of $\mathscr{F}$ to V, denoted $\mathscr{F}|_V$, is the pre-sheaf of abelian groups such that

$$\mathscr{F}|_V(U) = \mathscr{F}(U)$$

for each open set $U \subseteq Y$. In other words,

$$\mathscr{F}|_V(U) = \mathscr{F}(U' \cap Y)$$

for each open set $U \subseteq Y$ such that $U = U' \cap Y$. If $\mathscr{F}$ is a sheaf on X, it is clear that $\mathscr{F}|_V$ is also a sheaf on V.

Let's now consider a list of *abelian group like* operations on pre-sheaves of abelian groups on X. The key idea is that all *abelian group like* operations may be defined and verified 'open set by open set.' We start with a basic construction.

Definition 8.2. Let X be a topological space and let $\mathscr{F}$ be a pre-sheaf of abelian groups. A **sub pre-sheaf** of $\mathscr{F}$ is a pre-sheaf $\mathscr{F}'$ such that for every open set $U \subseteq X$, the assignment

$$U \mapsto \mathscr{F}'(U),$$

is such that $\mathscr{F}'(U)$ an abelian subgroup of $\mathscr{F}(U)$. It is easy to verify that this is indeed a pre-sheaf.

Definition 8.2 allows us to construct more instances of pre-sheafs.

Definition 8.3. Let X be a topological space and let $\mathscr{F}$ be a pre-sheaf of abelian groups. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of pre-sheafs.

(1) The **image pre-sheaf** of φ is a pre-sheaf such that for every open set $U \subseteq X$ the pre-sheaf is defined by the assignment

$$U \mapsto \mathrm{Im}(\varphi(U))$$

It is clear that this is is pre-sheaf by Definition 8.2.

(2) The **pre-sheaf kernel** of φ is a pre-sheaf such that for every open set $U \subseteq X$ the pre-sheaf is defined by the assignment

$$U \mapsto \ker(\varphi(U))$$

It is clear that this is is pre-sheaf by Definition 8.2. For a change of pace, we check that this is indeed the case by means of an diagram chasing argument. Let $U \subseteq X$ be an open set in X and let $V \subseteq U$. Then:

$$\varphi(V) \circ \mathrm{res}_{U,V} \circ \iota = \mathrm{res}_{U,V} \circ \underbrace{\varphi(U) \circ \iota}_{0} = 0.$$

By the universal property of $\ker(\varphi(U))$, we get a unique morphism μ

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi(V)) \arrow[r] & \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V) \\
0 \arrow[r] & \ker(\varphi(U)) \arrow[r, "\iota"] \arrow[u, dashed, "\mu"] & \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[u, "\mathrm{res}_{U,V}"] & \mathscr{G}(U) \arrow[u, "\mathrm{res}_{U,V}"]
\end{tikzcd}$$

The map μ serves as the restriction map. The other conditions are easy to check.

(3) Suppose φ is the inclusion morphism and $\mathscr{F}$ is a sub pre-sheaf of $\mathscr{G}$. The **quotient pre-sheaf** of $\mathscr{G}$ by $\mathscr{F}$ is a pre-sheaf such that for every open set $U \subseteq X$ pre-sheaf is defined by the assignment

$$U \mapsto \mathscr{G}(U)/\mathscr{F}(U)$$

It can be easily checked that this is a pre-sheaf.

(4) The **pre-sheaf cokernel** of φ is a pre-sheaf such that for every open set $U \subseteq X$ the pre-sheaf is defined by the assignment

$$U \mapsto \mathrm{coker}(\varphi(U))$$

This is a pre-sheaf by an argument as in (3).

In all constructions above, the restriction maps on the new pre-sheaf are induced by those of $\mathscr{F}$.

If $\mathscr{F}$ is a sheaf, then the image sheaf, the quotient sheaf, the sheaf kernel and the sheaf cokernel are defined to be the sheafification of the image pre-sheaf, the quotient pre-sheaf, the pre-sheaf kernel and the pre-sheaf cokernel. In general, we need to perform sheafification, as the next example shows.

Example 8.4. Consider:

$$U = \mathbb{C} \setminus \{0\} = \mathbb{C} \setminus [0, +\infty) \cup \mathbb{C} \setminus (-\infty, 0] = U_1 \cup U_2$$

Let exp be a sheaf morphism from $\mathscr{O}(U)$ to $\mathscr{O}^*(U)$ taking f to e^f. By complex analysis, the function $z \in \mathscr{O}^*(U)$ cannot be written as the exponential of some other holomorphic function $f \in \mathscr{O}(U)$. Thus $[z] \neq 0 \in \mathrm{coker}(\exp(U))$. On the other hand, the open sets U_1, U_2 are simply connected, so every nowhere zero function g can be written as $g = e^f$ on U_i. Thus $\mathrm{coker}(\exp(U_i)) = 0$ for $i = 1, 2$. This shows that the cokernel presheaf cannot be a sheaf, because the restriction of z to the open cover U_1, U_2 of U is zero on both sets, but it is globally non-zero.

However, we don't have to 'sheafifiy' in one special case as shown below:

Proposition 8.5. *Let X be a topological space and let $\mathscr{F}, \mathscr{G}$ be sheaves of R-mopdules on X. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a shaf morphism. The pre-sheaf kernel is a sheaf.*

Proof. Let $U \subseteq X$ be an open subset of X and let $\{U_i\}_{i\in I}$ be an open cover of U. Let $f \in \ker \varphi(U)$ such that $f|_{U_i} = 0$ for all $i \in I$. Since $\ker \varphi(U)$ is a R-submodule of $\mathscr{F}(U)$, this means that $f|_{U_i} = 0$ as a section of $\mathscr{F}(U)$, and so $f = 0$ by virtue of $\mathscr{F}$ being a sheaf. Suppose we have sections $f_i \in \ker \varphi(U_i) \subseteq \mathscr{F}(U_i)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all i and j. Since $\mathscr{F}$ is a sheaf, pick $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for all i. To see that f must be in $\ker \varphi(U)$, we need to show $\varphi(U)(f) = 0$. Pick $t \in \mathscr{G}(U)$ such that $t|_{U_i} = \varphi(U_i)(f_i) = 0$. Since φ commutes with the restriction maps, we have

$$\varphi(U)(f)|_{U_i} = \varphi(U_i)(f|_{U_i}) = \varphi(U_i)(f_i) = 0$$

Since $\mathscr{G}$ is a sheaf, we have that $\varphi(U)(f) = t = 0$. □

We end our discussion with two important constructions of sheaves that are related to the theory of vector bundles.

Definition 8.6. Let X be a topological space and let $\mathscr{F}, \mathscr{G}$ be pre-sheaves of R-modules. A **tensor product sheaf** of $\mathscr{F}$ and $\mathscr{G}$ is the sheafification of a pre-sheaf determined by the assignment

$$U \mapsto \mathscr{F}(U) \otimes_R \mathscr{G}(U)$$

for every open set $U \subseteq X$.

Proposition 8.7. *Suppose X be a smooth manifold and let $E, E' \to M$ be smooth vector bundles over X. We have*

$$\mathscr{E}_E \otimes_R \mathscr{E}_{E'} \cong \mathscr{E}_{E\otimes E'}$$

Here $\mathscr{E}$ is the sheaf of sections of the appropriate vector bundles.

Proof. Let $U \subseteq X$ be an open subset. A $\sigma \in \mathscr{E}_E(U) \otimes_R \mathscr{E}_{E'}(U)$ can be represented as a finite sum of abstract tensor products, $\sum_j \sigma_j \otimes \sigma'_j$. Let $\varphi(U)(\sigma)$ be the smooth section of the tensor product bundle $E \otimes E'$ over U given by the same formula:

$$x \mapsto \sum_j \sigma_j(x) \otimes \sigma'_j(x).$$

This gives a well-defined homomorphism

$$\varphi(U) : \mathscr{E}_E(U) \otimes_R (U)\mathscr{E}_{E'} \to \mathscr{E}_{E\otimes E'}(U),$$

Because $\varphi(V)$ is the restriction of $\varphi(U)$ whenever $V \subseteq U$, this defines a pre-sheaf morphism, F. The sheafification functor induces a sheaf morphism $\overline{F}$ from the $\mathscr{E}_E \otimes_R \mathscr{E}_{E'}$ to $\mathscr{E}_{E\otimes E'}$. We show that $\overline{F}$ is bijective. It is sufficient to show that $\overline{F}$ is bijective on stalks.

Let (s_j), (s'_k) be smooth local frames on U for E and E'. If $\gamma = \sum_{k,l} g^{kl} s_k \otimes s'_l$ is a smooth section of $E \otimes E'$ over U, then

$$\gamma = \varphi(U)\left(\sum_k s_k \otimes \left(\sum_l g^{kl} s'_l\right)\right)$$

So $\varphi(U)$ is surjective. Suppose $\sigma = \sum_j \sigma_j \otimes \sigma'_j$ is an element of $\mathscr{E}_E(U) \otimes_R \mathscr{E}_{E'}(U)$ such that $\varphi(U)(\sigma) = 0$. Write $\sigma_j = \sum_k f_j^k s_k$ and $\sigma'_j = \sum_l f_j'^l s'_l$ For all $x \in U$, we have

$$0 = \sum_j \sigma_j(x) \otimes \sigma'_j(x) = \sum_{j,k,l} f_j^k(x) f_j'^l(x) s_k(x) \otimes s'_l(x).$$

Since this is true for all $x \in U$ and the elements $s_k(x) \otimes s'_l(x)$ are linearly independent, it follows that $\sum_j f_j^k f_j'^l \equiv 0$ on U for each k and l. Thus

$$\sum_j \sigma_j \otimes \sigma'_j = \sum_k s_k \otimes \sum_l \left(\sum_j f_j^k f_j'^l\right) s'_l = 0,$$

so $\varphi(U)$ is injective. This proves the claim. □

Remark 8.8. *Clearly, Proposition 8.7 is true for complex manifolds as well (to be defined next).*

Part 3. Complex Manifolds

Complex manifolds are topological spaces locally modeled on open subsets in $\mathbb{C}^n$ with holomorphic transition functions. Complex manifolds are closely related to smooth manifolds, yet they exhibit notable distinctions in several aspects. The global counterparts of the similarities and differences between differentiable and holomorphic functions arise within the framework of complex manifold theory.

9. Definitions & Examples

9.1. Definitions. We first provide definitions.

Definition 9.1. Let X be a second-countable, Hausdorff, connected topological space. X is a **complex manifold of dimension** n if there is a there exists a collection $\{(U_i, \varphi_i)\}_{i\in I}$ such that:

1. U_i are open sets cover X,
2. Each $\varphi_i : U_i \to D_i$, where D_i is an open subset of some $\mathbb{C}^n$, is a homemorphism,
3. The transition functions
$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j),$$
are holomorphic.

Such a collection is called a **holomorphic atlas** for X. Each (U_i, ε_i) is called a **coordinate chart**.

Remark 9.2. *Since all holomorphic functions are smooth, a holomorphic atlas is also a smooth atlas and thus determines a unique smooth structure on X. Thus, every complex manifold is also a smooth manifold in a canonical way.*

Philosophically, a complex n-manifold locally resembles $\mathbb{C}^n$. This local resemblance enables the extension of many constructions valid in $\mathbb{C}^n$ to a complex manifold. For instance, this framework allows us to define holomorphic functions on X.

Definition 9.3. Let X be a complex manifold. A **holomorphic function** on X is a function $f : X \to \mathbb{C}$ such that $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ is holomorphic for some holomorphic chart (φ_i, U_i).

Remark 9.4. *It can be easily shown that if f is holomorphic, then f is holomorphic with respect to any holomorphic chart.*

We can now package all holomorphic functions on a complex manifold into the structure of a sheaf.

Definition 9.5. Let X be a complex manifold. The **structure sheaf on** X, denoted by $\mathscr{O}_X$, is the sheaf of holomorphic functions on X defined such that for any open subset $U \subseteq X$ we have

$$\mathscr{O}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ is a holomorphic function}\}.$$

Remark 9.6. *It is clear from the definition that via a holomorphic chart (U, φ) with $p \in U$ and $\varphi(p) = 0 \in \mathbb{C}^n$, the stalk $\mathscr{O}_{X,p}$ is isomorphic to $\mathscr{O}_{\mathbb{C}^n,0}$.*

We define holomorphic functions between complex manifolds analogously to how smooth functions are defined between smooth manifolds.

Definition 9.7. Let X and Y be complex manifolds. A **holomorphic map** from X to Y is a continuous function $F : X \to Y$ with the property that for every $p \in X$ there exist holomorphic coordinate charts (U, φ) for X and (V, ψ) for Y such that

- $p \in U$ and $F(p) \in V$
- $F(U) \subseteq V$
- The composite map $\psi \circ F \circ \varphi^{-1}$ is holomorphic as a map from $\varphi(U)$ to $\psi(V)$.

The function $\widehat{F} := \psi \circ F \circ \varphi^{-1}$ is called the coordinate representation of f with respect to the given holomorphic coordinates.

Remark 9.8. *If $F : X \to Y$ is a bijective holomorphic map with holomorphic inverse, then we say that F is a biholomorphism.*

The fundamental difference between complex and differentiable manifolds becomes manifest is given by the following proposition:

Proposition 9.9. *Let X be a compact connected complex manifold.*

(1) *Any global holomorphic function on X is constant. That is, $\mathscr{O}_X(X) \cong \mathbb{C}$.*
(2) *If $\dim X \geq 2$ and $p \in X$, then $\mathscr{O}_X(X) = \mathscr{O}_X(X \setminus \{p\}) = \mathbb{C}$.*

Proof. The proof is given below:

(1) Since X is compact, $|f| : X \to \mathbb{R}$ attains its maximum at some point $p \in X$. If (U_i, φ_i) is a holomorphic chart with $p \in U_i$, then $f \circ \varphi_i^{-1}$ is constant due to the maximum principle on $\varphi_i(U_i) \subseteq \mathbb{C}^n$. Hence, f is constant on U_i. Since X is connected, this shows that f must be constant[4]. Thus, $\mathscr{O}_X(X) \cong \mathbb{C}$.
(2) This follows from (1) and Proposition 1.13.

[4] A locally continuous functions on a connected space that is constant on an open set is constant.

This completes the proof. □

Here is another key difference between smooth manifolds and complex manifolds. A smooth manifold can always be covered by open subsets diffeomorphic to $\mathbb{R}^n$. In contrast, a complex manifold cannot be covered by open subsets biholomorphic to $\mathbb{C}^n$. This is because of the following proposition:

Proposition 9.10. *The unit ball $\mathbb{B}^{2n} \subseteq \mathbb{C}^n$ is not biholomorphic to $\mathbb{C}^n$.*

Proof. We know that $\mathbb{B}^{2n}$ and $\mathbb{C}^n$ are diffeomorphic. If $F : \mathbb{C}^n \to \mathbb{B}^{2n}$ is any holomorphic map, each of its coefficient functions is a bounded holomorphic function on $\mathbb{C}^n$ and therefore constant by Liouville's theorem. Thus, there is no biholomorphism between $\mathbb{B}^{2n}$ and $\mathbb{C}^n$. □

The definitions of topological manifolds, smooth manifolds, and complex manifolds have the same structure. Hence, we now introduce a convenient framework that includes smooth manifolds, complex manifolds, and many other kinds of spaces.

Definition 9.11. Let X be a connected topological space. A **geometric structure** on X, denoted as $\mathscr{G}$, is a sub-sheaf of the sheaf of continuous functions on X such that the abelian groups $\mathscr{G}(U) \subseteq \mathscr{C}(U)$ contain all constant functions for each open set $U \subseteq X$. The pair $(X, \mathscr{G})$ is called a **geometric space**.

We have already discussed various examples of geometric spaces. In order to view a complex manifold as a geometric space, we need to define the notion of a morphism of geometric spaces.

Definition 9.12. Let $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ be geometric spaces. A **morphism of geometric spaces** is a continuous map $f : X \to Y$ such that whenever $U \subseteq Y$ is open, and $g \in \mathscr{G}_Y(U)$, the composition $g \circ f$ belongs to $\mathscr{G}_X(f^{-1}(U))$.

Example 9.13. Let $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m$ and let $\mathscr{O}_X, \mathscr{O}_Y$ be the sheafs of holomorphic functions. A morphism $f : (\mathbb{C}^n, \mathscr{O}_X) \to (\mathbb{C}^m, \mathscr{O}_Y)$ is the same as a holomorphic mapping $f : \mathbb{C}^n \to \mathbb{C}^m$. This is because a continuous map $f : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if and only if it preserves complex-valued holomorphic functions. The forward direction is clear since a composition of holomorphic functions is holomorphic functions. The reverse direction follows easily since the hypothesis implies that each coordinate function of f is holomorphic.

Example 9.14. If $(X, \mathscr{G}_X)$ is a geometric space, then any open subset $U \subseteq X$ inherits a geometric structure $(U, \mathscr{G}_X|_U)$, where $\mathscr{G}_X|_U$ is the restriction sheaf. With this definition, the natural inclusion map $(U, \mathscr{G}|_U) \to (X, \mathscr{G}_X)$ becomes a morphism of geometric spaces.

Remark 9.15. *For a morphism $f : (X, \mathscr{G}_X) \to (Y, \mathscr{G}_Y)$ of geometric spaces, we typically write*

$$f_\# : \mathscr{G}_Y(U) \to \mathscr{G}_X(f^{-1}(U))$$

for the induced homomorphisms. We say that f is an isomorphism if it has an inverse that is also a morphism. This means that $f : X \to Y$ should be a homeomorphism, and that each map $f_\# : \mathscr{G}_Y(U) \to \mathscr{G}_X(f^{-1}(U))$ should be an isomorphism.

We can now give an alternative definition of a complex manifold.

Definition 9.16. A **complex manifold of dimension** n is a geometric space $(X, \mathscr{G}_X)$ such that each $p \in X$ has an open neighborhood $U \subseteq X$, such that $(U, \mathscr{G}_X|_U) \cong (D, \mathscr{O}|_D)$for some open subset $D \subseteq \mathbb{C}^n$.

Proposition 9.17. *Let X be a topological space. X is a complex manifold in the sense of Definition 9.16 if and only if X is a complex manifold in the sense of Definition 9.1.*

Proof. Let $(X, \mathscr{G}_X)$ be a complex manifold in the sense of Definition 9.16. We can find for each $p \in X$ an open neighborhood $p \in U_p$, together with an isomorphism of geometric spaces $\varphi_p : (U_p, \mathscr{G}_X|_{U_p}) \to (D_p, \mathscr{O}|_D)$, for $D_p \subseteq \mathbb{C}^n$ open. The transition maps are clearly biholomorphic. This defines a cordinates atlas on X. Conversely, let X be a complex manifold in the sense of Definition 9.1. Let $\{(U_i, \varphi_i)\}_{i \in I}$ be a holomorphic atlas. For $U \subseteq X$ open, set

$$\mathscr{G}_X(U) = \{f \in \mathscr{C}(U) \mid (f|_{U \cap U_i}) \circ \varphi_i^{-1} \text{ is holomorphic on } \varphi_i(U \cap U_i) \text{ for all } i \in I\}.$$

This makes sense because the transition functions are biholomorphic. It is easy to see that $\mathscr{O}_X$ is a sub-sheaf of $\mathscr{C}$. Hence, $(X, \mathscr{G}_X)$ is a geometric space. It is also a complex manifold, because every point has an open neighborhood (namely one of the U_i) that is isomorphic to an open subset of $\mathbb{C}^n$. □

Remark 9.18. *Is the dimension uniquely defined in Definition 9.1 or Definition 9.16? This is indeed the case. Let $x \in X$ such that x is contained in a coordinate chart (U, ϕ) such that $\phi(x) = 0$. Consider*

$$\mathscr{O}_x = \varinjlim_{x \in U} \mathscr{O}(U)$$

Let $\mathscr{O}_n$ denote the stalk at 0 of the sheaf of holomorphic functions on $\mathbb{C}^n$. Clearly,

$$\mathscr{O}_x \cong \mathscr{O}_n \cong \mathbb{C}[[z^1, \cdots, z^n]]$$

$\mathscr{O}_n$ is a local ring. Indeed, the unique maximal ideal is

$$\mathfrak{m}_n = \{f \in \mathscr{O}_n \mid f(0) = 0\};$$

If $f \in \mathscr{O}_n$ satisfies $f(0) \neq 0$, then f^{-1} is holomorphic in a neighborhood of the origin, and therefore $f^{-1} \in \mathscr{O}_n$. Clearly,

$$\mathscr{O}_n / \mathfrak{m}_n \cong \mathbb{C}$$

The integer n can be recovered from the ring $\mathscr{O}_n$. This is because Krull's dimension theory implies we have

$$\begin{aligned} n &= \dim(\mathbb{C}[[z^1, \ldots, z^n]]) \\ &= \dim\left(\mathbb{C}[[z^1, \ldots, z^n]]/\mathfrak{m}_n\right) + \operatorname{height}(\mathfrak{m}_n) \\ &= \dim(\mathbb{C}) + \operatorname{height}(\mathfrak{m}_n) \\ &= \operatorname{height}(\mathfrak{m}_n) \end{aligned}$$

This shows that $n = \dim X$ is well-defined.

Remark 9.19. *It follows that the function $x \mapsto \dim X$ is locally constant. Hence, if X is connected, the dimension is the same at each point, and the common value is called the dimension of the complex manifold X, denoted by $\dim X$, as we have done above.*

9.2. **Examples.** We now discuss several examples.

Example 9.20. $\mathbb{C}^n$ has a holomorphic structure determined by the holomorphic atlas consisting of the single coordinate chart $(\mathbb{C}^n, \mathrm{Id}_{\mathbb{C}^n})$. Similarly, the holomorphic structure on every open subset $U \subseteq \mathbb{C}^n$ is defined by the single chart (U, Id_U).

Example 9.21. (**Complex Projective Space**) For any $n \in \mathbb{N}$, the complex projective space, $\mathbb{CP}^n$ of dimension n is the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$, which we can identify with the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation defined by

$$w \sim w' \iff w' = \lambda w \text{ for some } \lambda \in \mathbb{C}^{\times}$$

We endow $\mathbb{CP}^n$ with the quotient topology. The equivalence class of $w \in \mathbb{C}^{n+1} \setminus \{0\}$ is denoted by $[w]$. Points of $\mathbb{CP}^n$ can be described through their homogeneous coordinates $[w^0, w^1, \dots, w^n]$. For each $\alpha = 0, \dots, n$, let $U_\alpha \subseteq \mathbb{CP}^n$ be the open subset $U_\alpha = \{[w] \in \mathbb{CP}^n : w^\alpha \neq 0\}$, and define a map

$$\begin{aligned} \varphi_\alpha \colon U_\alpha &\to \mathbb{C}^n \\ [w^0, \dots, w^n] &\mapsto \left(\frac{w^0}{w^\alpha}, \dots, \frac{w^{\alpha-1}}{w^\alpha}, \frac{w^{\alpha+1}}{w^\alpha}, \dots, \frac{w^n}{w^\alpha} \right) \end{aligned}$$

It is continuous by the characteristic property of the quotient topology, and it is a homeomorphism because it has a continuous inverse given by

$$\varphi_\alpha^{-1}(z^1, \dots, z^n) = [z^1, \dots, z^{\alpha-1}, 1, z^\alpha, \dots, z^n].$$

Thus each $(U_\alpha, \varphi_\alpha)$ is a coordinate chart, called affine coordinates for $\mathbb{CP}^n$. This makes $\mathbb{CP}^n$ into a topological manifold. For $\alpha < \beta$, the transition function between these charts can be computed explicitly as

$$\varphi_\alpha \circ \varphi_\beta^{-1}(z^1, \dots, z^n) = \left(\frac{z^1}{z^\alpha}, \dots, \widehat{\frac{z^\alpha}{z^\alpha}}, \dots, \frac{1}{z^\alpha}, \dots, \frac{z^n}{z^\alpha} \right),$$

where the hat indicates that the term in position α is omitted, and the $1/z^\alpha$ term is in position β. These transition functions are all holomorphic. It can be checked that $\mathbb{CP}^n$ is Haudorff and second-countable. Hence, $\mathbb{CP}^n$ is a complex manifold of dimension n.

Remark 9.22. *Note that $\mathbb{CP}^n$ is compact and connected because it is the image of the surjective continuous map*

$$q : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$$

given by $q(w^0, \dots, w^n) = [w^0, \dots, w^n]$, where $\mathbb{S}^{2n+1}$ is the set of unit vectors in $\mathbb{C}^{n+1}$. Moreover, we have,

$$\mathbb{CP}^n \cong \mathbb{S}^{2n+1}/\mathbb{S}^1$$

Remark 9.23. *Note that the quotient map $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is holomorphic. Indeed, if $w \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $w^\alpha \neq 0$, then let $U_\alpha \subseteq \mathbb{CP}^n$ and $q^{-1}(U_\alpha) \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ be open sets containing w and $[w]$ respectively. Clearly, $q(q^{-1}(U_\alpha)) \subseteq U_\alpha$. Moreover, we have,*

$$\varphi_\alpha \circ q \circ \mathrm{Id}^{-1}_{\mathbb{C}^{n+1}\setminus\{0\}}(z^0, \cdots, z^n) = \varphi_\alpha[z^0, \cdots, z^n] = \left(\frac{z^0}{z^\alpha}, \dots, \frac{z^{\alpha-1}}{z^\alpha}, \frac{z^{\alpha+1}}{z^\alpha}, \dots, \frac{z^n}{z^\alpha} \right).$$

which is holomorphic. Hence, q is holomorphic. Therefore, if $f : \mathbb{CP}^n \to \mathbb{C}$ is any holomorphic function, then $g_f := f \circ q$ is also a holomorphic function such that g_f is scale-invariant, i.e, $g_f(\lambda \cdot) = g_f(\cdot)$ for each $\lambda \in \mathbb{C}^\times$. In fact, the converse is true as well. Let $f : \mathbb{CP}^n \to \mathbb{C}$ be a continuous such that $g_f := f \circ q$ is holomorphic and g_f is scale-invariant. Then if

(U_β, φ_β) is a chart on $\mathbb{CP}^n$, then,

$$\begin{aligned} f \circ \varphi_\beta^{-1}(z^1, \cdots, z^n) &= f[z^1, \cdots, z^{\alpha-1}, 1, z^{\alpha+1}, \cdots, z^n] \\ &= f \circ q(z^1, \cdots, z^{\alpha-1}, 1, z^{\alpha+1}, \cdots, z^n) \\ &= g_f(z^1, \cdots, z^{\alpha-1}, 1, z^{\alpha+1}, \cdots, z^n) \end{aligned}$$

is holomorphic. Therefore we have that the corresponding sheaf on $\mathbb{CP}^n$ is given by

$$\begin{aligned} \mathscr{G}_X(U) &= \{f \in \mathscr{C}(U) \mid (f|_{U \cap U_i}) \circ \varphi_i^{-1} \text{ is holomorphic on } \varphi_i(U \cap U_i) \text{ for all } i \in I\} \\ &= \{f \in \mathscr{C}(U) \mid g_f = f \circ q \text{ is holomorphic and } g_f \text{ is scale invariant}\} \end{aligned}$$

Example 9.24. (**Complex Lie Groups**) A complex Lie group, G, is a complex manifold that is also a group such that the map $(x, y) \mapsto x \cdot y^{-1}$ is holomorphic. Examples of complex Lie groups are provided by $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, and $\mathrm{Sp}(n, \mathbb{C})$. They are certainly not abelian for $n > 1$.

Remark 9.25. *Note that certain classical groups like $\mathrm{U}(n, \mathbb{C})$ are often not complex complex Lie groups, but just ordinary real Lie groups. The easiest way of proving that $\mathrm{U}(n, \mathbb{C})$ is not a complex Lie group consists in using the fact that its Lie algebra $\mathfrak{u}(n)$ is not a complex Lie algebra. We have*

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) \mid A = -A^*\}$$

Unless $A = 0$, if $A \in \mathfrak{u}(n)$, then $iA \notin \mathfrak{u}(n)$.

Another method to construct complex manifolds is to consider the quotient space when a group acts by automorphisms on the complex manifold. The next two propositions allow us to construct new complex manifolds in this manner.

Proposition 9.26. *Let Y be a connected complex manifold and $\pi\colon X \to Y$ be a covering map. Then X is a complex manifold and has a unique holomorphic atlas such that π is a holomorphic covering map.*

Proof. (**Sketch**) We know from smooth manifold theory that X is a topological manifold and has a unique smooth structure such that π is a smooth covering map. We can define holomorphic charts on X as follows: Given a point $p \in X$, let U be an evenly covered neighborhood of $\pi(p)$. After shrinking U if necessary, we can find a holomorphic coordinate map $\varphi\colon U \to \mathbb{C}^n$. Let $\widetilde{U}$ be the connected component of $\pi^{-1}(U)$ containing p, and define $\widetilde{\varphi} = \varphi \circ \pi\colon \widetilde{U} \to \mathbb{C}^n$. When two such charts $(U, \widetilde{\varphi})$ and $(V, \widetilde{\psi})$ overlap, in a neighborhood of each point the transition function can be expressed as

$$\widetilde{\psi}^{-1} \circ \widetilde{\varphi} = \psi^{-1} \circ \varphi,$$

which in this case is holomorphic. Clearly, this makes π into a local biholomorphism. □

Proposition 9.27. *(**Holomorphic Quotient Manifold Theorem**) Let X be a complex manifold and Γ is a discrete (complex) Lie group acting holomorphically[5], freely[6], and properly[7] on X. Then the quotient space X/Γ has a unique complex manifold structure such that the quotient map $q\colon X \to X/\Gamma$ is a holomorphic (normal) covering map.*

[5]The action Γ on X is holomorphic if the map $x \mapsto g \cdot x$ is holomorphic for each $g \in \Gamma$.

[6]An action of Γ on X is free if $g \cdot x = x$ for some $g \in \Gamma$ and $x \in X$ implies g is the identity.

[7]The action of Γ on X is proper if the map $\Gamma \times X \to X \times X$ given by $(g, x) \mapsto (g \cdot x, x)$ is a proper map.

Proof. (**Sketch**) We know from smooth manifold theory that that X/Γ has a unique smooth manifold structure such that q is a smooth (normal) covering map. To define a complex manifold structure on X/Γ, let $U \subseteq X/\Gamma$ be any evenly covered open set, and choose a smooth local section $\sigma\colon U \to X$. Because X is a complex manifold, $\sigma(U)$ has a covering by holomorphic charts $(U_\alpha, \varphi_\alpha)$, and for each such chart we can define $(\sigma^{-1}(U_\alpha), \varphi_\alpha \circ \sigma)$ as a chart for X/Γ. For a fixed local section σ, all of these charts are holomorphically compatible with each other. If $\tilde{\sigma}\colon U \to X$ is any other local section, there is an element $g \in \Gamma$ such that $\tilde{\sigma}(x) = g \cdot \sigma(x)$ for all $x \in U$; and the fact that $x \mapsto g \cdot x$ is a biholomorphism of X with inverse $x \mapsto g^{-1} \cdot x$ guarantees that the charts obtained from $\tilde{\sigma}$ will be holomorphically compatible with those obtained from σ. □

Corollary 9.28. *Suppose G is a connected complex Lie group and $\Gamma \subseteq G$ is a discrete subgroup.The left coset space G/Γ is a complex manifold, and the quotient map $\pi\colon G \to G/\Gamma$ is a holomorphic (normal) covering map.*

Proof. This follows from Proposition 9.27 since the action automatically satisfies assumptions in Proposition 9.27. □

Remark 9.29. *It is clear that the quotient maps in Proposition 9.26, Proposition 9.27 and Corollary 9.28 are local biholomorphisms. An argument similar to that in Remark 9.23 shows that,*

$$\mathscr{O}_{G/\Gamma}(U) = \left\{ f \in \mathscr{O}_G\left(q^{-1}(U)\right) \mid f \circ \gamma = f \text{ for every } \gamma \in \Gamma \right\}.$$

Example 9.30. (**Complex Tori**) Suppose V is an n-dimensional complex vector space, considered as an abelian complex Lie group. A lattice $\Lambda \subseteq V$ is a subgroup $\Lambda \subseteq V$ generated by taking $\mathbb{Z}$-linear combinations of $2n$ $\mathbb{R}$-linearly independent vectors $v_1, \ldots, v_{2n}$. By Corollary 9.28 V/Λ is an n-dimensional complex Lie group, called a complex torus.

Remark 9.31. *When $n = 0$, V/Λ is a single point. When $n > 0$, we can think of V as a $2n$ real-vector space with the the real-linear isomorphism*

$$A\colon \mathbb{R}^{2n} \to V, \quad A(x^1, \ldots, x^{2n}) = \sum_{j=1}^{2n} x^j v_j$$

This map descends to a map $\widetilde{A}$:

$$\begin{tikzcd} \mathbb{R}^{2n} \arrow[r, "A"] \arrow[d, two heads, "q"'] & V \arrow[r, two heads] & V/\Lambda \\ \mathbb{R}^{2n}/\mathbb{Z}^{2n} \arrow[urr, dotted, "\widetilde{A}"'] & & \end{tikzcd}$$

Since q is a smooth covering map and hence a smooth submersion and the map $\mathbb{R}^{2n} \to V/\Lambda$ is smooth, we have that $\widetilde{A}$ is a smooth map. Since $\widetilde{A}$ is bijective and a local diffeomorphism (because the maps q and $\mathbb{R}^{2n} \to V/\Lambda$ are local diffeomorphisms) , we have

$$V/\Lambda \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong (\mathbb{R}/\mathbb{Z})^{2n} \cong \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{2n \text{ times}}$$

as smooth manifolds. Thus, the complex tori defined by different lattices are all diffeomorphic to each other. Moreover, this argument also shows that the complex tori are compact connected smooth manifolds.

Example 9.32. (**Hopf Manifold**) There is a diffeomorphism

$$\varphi\colon \mathbb{S}^{2n-1} \times \mathbb{R} \to \mathbb{C}^n \setminus \{0\}$$

given by $\varphi(z^1, \cdots, z^n, t) = (e^t z^1, \cdots, e^t z^n)$. Let $\mathbb{Z}$ naturally acts on $\mathbb{S}^{2n-1} \times \mathbb{R}$, by letting

$$m \cdot (z^1, \cdots, z^n, t) = (z^1, \cdots, z^n, t + m)$$

for $m \in \mathbb{Z}$. Clearly, the resulting quotient space is diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. The diffeomorphism φ allows us to transfer the action of $\mathbb{Z}$ to an action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$. Explicitly, it is given by the formula

$$m \cdot (z^1, \cdots, z^n) = (e^m z^n, \cdots, e^m z^n)$$

$\mathbb{Z}$ acts by biholomorphisms and the action is clearly free and properly discontinuous. Hence, $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}$ is a complex manifold called the Hopf manifold. By construction, it is diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ (and hence compact).

Remark 9.33. *A general version of Corollary 9.28 holds. If G is a complex Lie group and H is a (closed) complex Lie subgroup of G acting on G holomorphically, freely, and properly, then the quotient G/H is a complex manifold, and the quotient map $\pi : G \to G/H$ is holomorphic.*

Example 9.34. (**Complex Grassmanian**) Let $\mathrm{Gr}_k(\mathbb{C}^n)$ denote the set of k-dimensional subspaces in $\mathbb{C}^n$. $\mathrm{GL}(n, \mathbb{C})$ acts transitively on $\mathrm{Gr}_k(\mathbb{C}^n)$. Indeed, if $U, U' \in \mathrm{Gr}_k(\mathbb{C}^n)$, we can let $\mathscr{B}$ and $\mathscr{B}'$ be basis for $\mathbb{C}^n$ obtained by extending a basis for U and U' respectively. A change of basis matrix from $\mathscr{B}$ to $\mathscr{B}'$ then maps U to U'. The isotropy subgroup of $U = \langle e_1, \ldots, e_k \rangle$ is

$$H = \begin{pmatrix} *_k & * \\ 0 & *_{n-k} \end{pmatrix}$$

Here $*_k$ is a k by k matrix and $*_{n-k}$ is a $(n-k)$ by $(n-k)$ matrix. Thus $\mathrm{Gr}_k(\mathbb{C}^n)$ is the coset space $\mathrm{GL}(n, \mathbb{C})/H$. Both $\mathrm{GL}(n, \mathbb{C})$ and H are complex Lie groups (open subsets of some $\mathbb{C}^N$). Hence, $\mathrm{Gr}_k(\mathbb{C}^n)$ is a complex manifold by Remark 9.33.

Remark 9.35. *The Grassmanian is compact. Indeed, observing that we can choose orthonormal bases of subspaces, we have that $\mathrm{U}(n, \mathbb{C})$ acts continuously and transitively on $\mathrm{G}_k(\mathbb{C}^n)$ and we have*

$$\mathrm{Gr}_k(\mathbb{C}^n) = \frac{\mathrm{U}(n, \mathbb{C})}{\mathrm{U}(k, \mathbb{C}) \times \mathrm{U}(n-k, \mathbb{C})}$$

Since $\mathrm{U}(m, \mathbb{C})$ is compact for all $m \in \mathbb{N}$, we have that $\mathrm{Gr}_k(\mathbb{C}^n)$ is compact.

Remark 9.36. *Another important class of complex manifolds is that of holomorphic vector bundles. The general theory of holomorphic vector bundles is more or less the same as for smooth vector bundles. Smooth vector bundles are discussed in the Riemannian geometry notes. The language of (holomorphic) vector bundles will be used throughout the rest of the notes.*

10. Tangent Vectors & Tangent Bundle

10.1. **Tangent Vectors.** Recall that on smooth manifolds, we can make sense of calculus by introducing the tangent space at a point, which serves as a 'linear model' for the manifold near that point. We now extend this concept to the case of complex manifolds. If $X = \mathbb{R}^n$, $T_p\mathbb{R}^n$ denote the tangent space at $p \in \mathbb{R}^n$. Recall that, we have

$$T_p\mathbb{R}^n \cong \mathrm{Der}_p\mathbb{R}^n,$$

where $\mathrm{Der}_p\mathbb{R}^n$ is the space of linear operators $D_p : C^\infty_{\mathbb{R}}(\mathbb{R}^n) \to \mathbb{R}$ with the property

$$D_p(fg) = f(p)D_p(g) + g(p)D_p(f)$$

for all $f, g \in C^\infty_{\mathbb{R}}(\mathbb{R}^n)$. Here $\mathscr{C}^\infty_{\mathbb{R}}(\mathbb{R}^n)$ denotes sheaf of $\mathbb{R}$-valued smooth functions on $\mathbb{R}^n$. It suffices to replace $\mathscr{C}^\infty_{\mathbb{R}}(\mathbb{R}^n)$ by $\mathscr{C}^\infty_{p,\mathbb{R}}$ in the definition of $\mathrm{Der}_p\mathbb{R}^n$ since $\mathrm{Der}_p\mathbb{R}^n$ is a local operator. If X is a smooth manifold, we have

$$T_pX = \mathrm{Der}_pX$$

If X is a complex manifold we can now use the complexification functor to define the tangent space to a point $p \in X$. Indeed, we make the following definition.

Definition 10.1. Let X be a complex manifold. The **complex tangent space** at $p \in X$ is given by $(T_pX)^{\mathbb{C}} := T_pX \otimes_{\mathbb{R}} \mathbb{C}$

We set $\mathrm{Der}^{\mathbb{C}}_p(X) \cong \mathrm{Der}_p(X) \otimes_{\mathbb{R}} \mathbb{C}$. What is $\mathrm{Der}^{\mathbb{C}}_p(X)$ concretely? This is just the complexification of $\mathrm{Der}_p(X)$. Morally, we can think of elements of $\mathrm{Der}^{\mathbb{C}}_p(X)$ as the set of all linear combinations of elements of $\mathrm{Der}_p(X)$ with complex coefficients. Hence, elements of $\mathrm{Der}_p(X)$ are of the form $X_p + iY_p$ such that $X_p, Y_p \in \mathrm{Der}_p(X)$. We can think of elements of $\mathrm{Der}^{\mathbb{C}}_p(X)$ as $\mathbb{C}$-linear derivations on $\mathscr{C}^\infty_{p,\mathbb{C}}$, where $\mathscr{C}^\infty_{p,\mathbb{C}}$ is the stalk of the sheaf of complex-valued *smooth* functions on X, denoted as $\mathscr{C}^\infty_{\mathbb{C}}$. We have

$$\begin{aligned}(T_pX)^{\mathbb{C}} &= T_pX \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong \mathrm{Der}_p(X) \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong \mathrm{Der}^{\mathbb{C}}_p(X)\end{aligned}$$

If X, Y be smooth manifolds and $F : X \to Y$ is a smooth map, recall that for $p \in X$, the differential of F is the map

$$dF_p : T_pX \to (T_{F(p)}Y)^{\mathbb{R}}$$

defined by $dF_p(D_p)(f) = D_p(f \circ F)$ for $D_p \in T_pX$ and $f \in C^\infty_{\mathbb{R}}(Y)$. This leads to the following definition.

Definition 10.2. Let X and Y be complex manifolds and let $F : X \to Y$ be a holomorphic map. The complex differential of F at p - denoted as $(dF_p)^{\mathbb{C}}$ - is the complexification of the linear map dF_p.

10.2. **Computations in Coordinates.** We discuss how to do computations with tangent vectors in local coordinates. We first discuss the smooth manifold case. Suppose X is a n-dimensional smooth manifold and let (U, ϕ) be a smooth coordinate chart on X. Write the local coordinate as $(x^1, \cdots, x^n)$. Recall that $d\phi_p : T_pX \to T_{\phi(p)}\mathbb{R}^n$ is an isomorphism. The derivations

$$\left.\frac{\partial^{\mathbb{R}^n}}{\partial x^1}\right|_{\phi(p)}, \cdots, \left.\frac{\partial^{\mathbb{R}^n}}{\partial x^n}\right|_{\phi(p)}$$

form a basis for $T_p\mathbb{R}^n$. Therefore, the preimages of these vectors under the isomorphism $d\phi_p$ form a basis for T_pX. We write these basis vectors as $\frac{\partial^X}{\partial x^1}|_p, \cdots, \frac{\partial^X}{\partial x^n}|_p$ and we have:

$$\begin{aligned}\left.\frac{\partial^X}{\partial x^i}\right|_p &= (d\phi_p)^{-1}\left(\left.\frac{\partial^{\mathbb{R}^n}}{\partial x^i}\right|_{\phi(p)}\right)\\ &= (d\phi^{-1})_{\phi(p)}\left(\left.\frac{\partial^{\mathbb{R}^n}}{\partial x^i}\right|_{\phi(p)}\right)\end{aligned}$$

We see that $\frac{\partial^X}{\partial x^i}|_p$ acts on a function $f \in C^\infty_{\mathbb{R}}(U)$ by

$$\begin{aligned}\left.\frac{\partial^X f}{\partial x^i}\right|_p &= (d\phi^{-1})_{\phi(p)}\left(\left.\frac{\partial^{\mathbb{R}^n}}{\partial x^i}\right|_{\phi(p)}\right)f\\ &= \left.\frac{\partial^{\mathbb{R}^n}(f\circ\phi^{-1})}{\partial x^1}\right|_{\phi(p)}\\ &:= \left.\frac{\partial^{\mathbb{R}^n}(\hat{f})}{\partial x^1}\right|_{\phi(p)}\end{aligned}$$

where $\hat{f} = f \circ \phi^{-1}$ is the coordinate representation of f. In other words, $\frac{\partial^X}{\partial x^i}|_p$ is just the derivation that takes the ith partial derivative of the coordinate representation of f at the coordinate representation of p. It is important to note that these computations depend on the chart ϕ of choice. A tangent vector $D_p \in T_pM$ can be written uniquely as a linear combination

$$D_p = D_p^i \left.\frac{\partial^X}{\partial x^i}\right|_p$$

Now let X be a complex manifold of dimension n. Then X can be thought of as a smooth manifold of dimension $2n$. Let (U, ϕ) be a holomorphic coordinate chart. Write local coordinates as $(z^1, \cdots, z^n)$. We have $z^i = x^i + iy^i$. The (real) tangent space at the point p is:

$$(T_pX)^{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right\rangle.$$

The vector fields $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ are interpreted as smooth vector fields on $U \subseteq X$. These vector fields are called complex coordinate vector fields. The complexified tangent space is

$$\begin{aligned}(T_pX)^{\mathbb{C}} &= \mathrm{Span}_{\mathbb{C}}\left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right\rangle\\ &= \mathrm{Span}_{\mathbb{C}}\left\langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n}\right\rangle.\end{aligned}$$

Here the alternative basis in the second line is again given by

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

Remark 10.3. *If $X = \mathbb{C}^n$ and f is a holomorphic function on $\mathbb{C}^n$, recall that*

$$\frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)f = \frac{1}{2}\left(\frac{\partial f}{\partial x^j} - i\frac{\partial f}{\partial y^j}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial z^j} + \frac{\partial f}{\partial z^j}\right) = \frac{\partial f}{\partial z^j}$$

This motivates the consideration of the alternative basis for $(T_pX)^{\mathbb{C}}$.

This define a smooth local complex frame $\left\{\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j}\right\}_{j=1}^n$ for $(T_pX)^{\mathbb{C}}$. Consider the following two subspaces:

$$(T_pX)_{(1,0)} = \mathbb{C}\left\langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right\rangle, \qquad (T_pX)_{(0,1)} = \mathbb{C}\left\langle \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} \right\rangle$$

It can be easily checked that $T^{\mathbb{C}}_{p,\pm 1}X$ is the $\pm i$ eigenspace of the complex structure on $(T_pX)^{\mathbb{C}}$ induced by multiplication by i. Therefore, Proposition 2.9 implies that we have a direct sum decomposition

$$(T_pX)^{\mathbb{C}} = (T_pX)_{(1,0)} \oplus (T_pX)_{(0,1)}$$

Remark 10.4. *$(T_pX)^{\mathbb{C}}_{(1,0)}$ is called the holomorphic tangent space and $(T_pX)^{\mathbb{C}}_{(0,1)}$ is called the anti-holomorphic tangent space.*

Remark 10.5. *If $f\colon X \to \mathbb{C}$ is a smooth function in a co-ordinate chart $(z^1, \dots, z^n)$, then f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}^j} \equiv 0$ on U for $j = 1, \dots, n$. This follows readily from Remark 1.9.*

We now discuss how to compute the differential of a smooth map between complex manifolds in coordinates.

Proposition 10.6. *Let X and Y be complex manifolds and $F\colon X \to Y$ be a holomorphic map. Let $p \in X$ and let $(dF_p)^{\mathbb{C}}$ be the complexified differential from $(T_pX)^{\mathbb{C}}$ to $(T_pY)^{\mathbb{C}}$. Let $z^j = x^j + iy^j$ be local holomorphic coordinates for X in a neighborhood of p, and $w^j = u^j + iv^j$ for N in a neighborhood of $F(p)$. We have:*

$$(dF_p)^{\mathbb{C}}\left(\left.\frac{\partial}{\partial z^j}\right|_p\right) = \frac{\partial F^k}{\partial z^j}(p)\left.\frac{\partial}{\partial w^k}\right|_{F(p)} + \frac{\partial \overline{F}^k}{\partial \bar{z}^j}(p)\left.\frac{\partial}{\partial \bar{w}^k}\right|_{F(p)},$$
$$(dF_p)^{\mathbb{C}}\left(\left.\frac{\partial}{\partial \bar{z}^j}\right|_p\right) = \frac{\partial F^k}{\partial \bar{z}^j}(p)\left.\frac{\partial}{\partial w^k}\right|_{F(p)} + \frac{\partial \overline{F}^k}{\partial \bar{z}^j}(p)\left.\frac{\partial}{\partial \bar{w}^k}\right|_{F(p)}.$$

Proof. (Sketch) We use the standard coordinate formula for the differential of a smooth map between smooth manifolds without providing a proof. Write F as $F = U + iV$. Considering X and Y as smooth manifolds, we have the usual coordinate formula for dF_p:

$$dF_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \frac{\partial U^k}{\partial x^j}(p)\left.\frac{\partial}{\partial u^k}\right|_{F(p)} + \frac{\partial V^k}{\partial x^j}(p)\left.\frac{\partial}{\partial v^k}\right|_{F(p)},$$
$$dF_p\left(\left.\frac{\partial}{\partial y^j}\right|_p\right) = \frac{\partial U^k}{\partial y^j}(p)\left.\frac{\partial}{\partial u^k}\right|_{F(p)} + \frac{\partial V^k}{\partial y^j}(p)\left.\frac{\partial}{\partial v^k}\right|_{F(p)}.$$

We transform this formula into holomorphic coordinates. Using the definitions of $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \bar{z}^j}$, we obtain:

$$(dF_p)^{\mathbb{C}}\left(\left.\frac{\partial}{\partial z^j}\right|_p\right) = \frac{\partial U^k}{\partial z^j}(p)\left.\frac{\partial}{\partial u^k}\right|_{F(p)} + \frac{\partial V^k}{\partial z^j}(p)\left.\frac{\partial}{\partial v^k}\right|_{F(p)},$$
$$(dF_p)^{\mathbb{C}}\left(\left.\frac{\partial}{\partial \bar{z}^j}\right|_p\right) = \frac{\partial U^k}{\partial \bar{z}^j}(p)\left.\frac{\partial}{\partial u^k}\right|_{F(p)} + \frac{\partial V^k}{\partial \bar{z}^j}(p)\left.\frac{\partial}{\partial v^k}\right|_{F(p)}.$$

Now substitute $\frac{\partial}{\partial u^k} = \frac{\partial}{\partial w^k} + \frac{\partial}{\partial \overline{w}^k}$ and $\frac{\partial}{\partial v^k} = i\left(\frac{\partial}{\partial w^k} - \frac{\partial}{\partial \overline{w}^k}\right)$ and collect terms:

$$(dF_p)^{\mathbb{C}}\left(\left.\frac{\partial}{\partial z^j}\right|_p\right) = \left(\frac{\partial U^k}{\partial z^j}(p) + i\frac{\partial V^k}{\partial z^j}(p)\right)\left.\frac{\partial}{\partial w^k}\right|_{F(p)} + \left(\frac{\partial U^k}{\partial z^j}(p) - i\frac{\partial V^k}{\partial z^j}(p)\right)\left.\frac{\partial}{\partial \bar{w}^k}\right|_{F(p)},$$

$$(dF_p)^{\mathbb{C}}\left(\left.\frac{\partial}{\partial \bar{z}^j}\right|_p\right) = \left(\frac{\partial U^k}{\partial \bar{z}^j}(p) + i\frac{\partial V^k}{\partial \bar{z}^j}(p)\right)\left.\frac{\partial}{\partial w^k}\right|_{F(p)} + \left(\frac{\partial U^k}{\partial \bar{z}^j}(p) - i\frac{\partial V^k}{\partial \bar{z}^j}(p)\right)\left.\frac{\partial}{\partial \bar{w}^k}\right|_{F(p)}.$$

The desired formulas now follow. □

Corollary 10.7. ***(Chain Rule in Coordinates)*** *Let X and Y be complex manifolds and let $F\colon X \to Y$ is a holomorphic map, and $h\colon Y \to \mathbb{C}$ is a holomorphic function. In terms of local holomorphic coordinates (z^j) for M and (w^k) for N, we have:*

$$\frac{\partial(h \circ F)}{\partial z^j} = \frac{\partial h}{\partial w^k}\frac{\partial F^k}{\partial z^j} + \frac{\partial h}{\partial \bar{w}^k}\frac{\partial \overline{F}^k}{\partial z^j},$$
$$\frac{\partial(h \circ F)}{\partial \bar{z}^j} = \frac{\partial h}{\partial w^k}\frac{\partial F^k}{\partial \bar{z}^j} + \frac{\partial h}{\partial \bar{w}^k}\frac{\partial \overline{F}^k}{\partial \bar{z}^j}.$$

Proof. This follows from Proposition 10.6 upon noting that the value of $\frac{\partial(h\circ F)}{\partial z^j}$ at $p \in X$ is equal to the $\frac{\partial}{\partial w}$ component of $(d(h \circ F)_p)^{\mathbb{C}}(\frac{\partial}{\partial z^j}|_p)$. But this expression is $(dh_{F(p)} \circ dF_p)^{\mathbb{C}}(\frac{\partial}{\partial z^j}|_p)$. The formula for $\frac{\partial(h\circ F)}{\partial z^j}$ at $p \in X$ now follows by invoking the formulas in Proposition 10.6. A similar argument applies to the $\frac{\partial(h\circ F)}{\partial \bar{z}^j}$ derivative $p \in X$. □

10.3. **Tangent Bundle.** We can now construct the tangent bundle associated with a complex manifold. Let X be a complex n-manifold. Let $\pi\colon TX \to X$ be the smooth rank-$2n$ tangent bundle over the underlying smooth manifold structure on X. Define the complexification of TX to be the set

$$(TX)^{\mathbb{C}} = \coprod_{p\in X}(T_pX)^{\mathbb{C}}$$

together with the obvious projection $\pi_{\mathbb{C}}\colon (TX)^{\mathbb{C}} \to X$. For each smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^{2n}$, we define a local trivialization $\Phi^{\mathbb{C}}\colon (\pi^{-1})^{\mathbb{C}}(U) \to U \times \mathbb{C}^{2n}$ by

$$\Phi^{\mathbb{C}}(\xi) = \big(\pi^{\mathbb{C}}(\xi), (\Phi|_{TX_{\pi^{\mathbb{C}}(\xi)}})^{\mathbb{C}}(\xi)\big).$$

Wherever two such trivializations (U, Φ) and (V, Ψ) overlap, we can write

$$\Psi \circ \Phi^{-1}(p, v) = (p, \tau(p)v)$$

for some smooth transition function $\tau\colon U \cap V \to \mathrm{GL}(2n, \mathbb{R})$. Clearly the transition function from $\Phi_{\mathbb{C}}$ to $\Psi_{\mathbb{C}}$ is the same:

$$\Psi^{\mathbb{C}} \circ (\Phi^{-1})^{\mathbb{C}}(p, v) = (p, \tau(p)v),$$

where now τ is considered as a map into $\mathrm{GL}(2n, \mathbb{C})$. It follows from the vector bundle chart lemma [Lee12] (adapted in the obvious way for holomorphic vector bundles) that $\pi^{\mathbb{C}}\colon (TX)^{\mathbb{C}} \to X$ has a unique structure as a smooth rank-$2n$ holomorphic vector bundle, with the maps constructed above as smooth local trivializations.

Definition 10.8. Let X be a complex manifold. A **holomorphic vector field** is a section of $(TX)^{\mathbb{C}}$.

A holomorphic vector field can be written as $Z = X + iY$, where X, Y are smooth vector fields. Z acts on a holomorphic function $f = u + iv$ by

$$Zf = Xf + iYf = (Xu + iXv + Yu + iYv) = (Xu + Yu) + i(Xv + Yv)$$

The Lie bracket operation can be extended to pairs of smooth holomorphic vector fields by complex bilinearity:

$$[X_1 + iY_1, X_2 + iY_2] = ([X_1, X_2] - [Y_1, Y_2]) + i([X_1, Y_2] + [Y_2, X_1]).$$

Example 10.9. Let $X = \mathbb{C}^n$. We have the following facts:

(1) We have

$$T\mathbb{C}^n = \coprod_{p\in\mathbb{C}^n} T_p\mathbb{C}^n = \coprod_{p\in\mathbb{C}^n} (\mathbb{C}^n)_{\mathbb{R}} = (\mathbb{C}^n)_{\mathbb{R}} \times \mathbb{C}^n,$$
$$(T\mathbb{C}^n)^{\mathbb{C}} = \coprod_{p\in\mathbb{C}^n} (T_p\mathbb{C}^n)^{\mathbb{C}} = \coprod_{p\in\mathbb{C}^n} \mathbb{C}^n = \mathbb{C}^n \times \mathbb{C}^n.$$

(2) The complexified tangent bundle $T^{\mathbb{C}}\mathbb{C}^n$ splits as:

$$\begin{aligned}(T\mathbb{C}^n)^{\mathbb{C}} &= T_{(1,0)}\mathbb{C}^n \oplus T_{(0,1)}\mathbb{C}^n \\ &= \mathrm{Span}_{\mathbb{C}} \left\langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right\rangle \oplus \mathrm{Span}_{\mathbb{C}} \left\langle \frac{\partial}{\partial \overline{z}^1}, \dots, \frac{\partial}{\partial \overline{z}^n} \right\rangle.\end{aligned}$$

(3) $T\mathbb{C}^n$ has a canonical complex structure $J_{\mathbb{C}^n}$, which satisfies:

$$J_{\mathbb{C}^n}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \quad J_{\mathbb{C}^n}\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}.$$

(4) For an open subset $U \subseteq \mathbb{C}^n$, a smooth function $F\colon U \to \mathbb{C}^m$ is holomorphic if and only if the following relation holds for all $p \in U$:

$$DF(p) \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ DF(p).$$

for all $p \in U$. This follows from the following computation:

$$\begin{aligned} DF\left(J_{\mathbb{C}^n}\frac{\partial}{\partial \overline{z}^j}\right) - J_{\mathbb{C}^m}\left(DF\frac{\partial}{\partial \overline{z}^j}\right) &= DF\left(-i\frac{\partial}{\partial \overline{z}^j}\right) - J_{\mathbb{C}^m}\left(DF\frac{\partial}{\partial \overline{z}^j}\right) \\ &= -i\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k} - i\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{w}^k} - J_{\mathbb{C}^m}\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k} - J_{\mathbb{C}^m}\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{w}^k} \\ &= -2i\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k}.\end{aligned}$$

If X is an arbitrary complex manifold, we now argue that $(TX)^{\mathbb{C}}$ has a canonical complex structure that induces a canonical decomposition of $(TX)^{\mathbb{C}}$:

Proposition 10.10. *Let X be a complex n-manifold.*

(1) *There is a canonical complex structure on TX, denoted by $J_X : (TX)^{\mathbb{C}} \to (TX)^{\mathbb{C}}$.*
(2) *There are smooth sub-bundles $(TX)_{(1,0)}, (TX)_{(0,1)} \subseteq (TX)^{\mathbb{C}}$ whose fibers at each point are the i-eigenspace and $(-i)$-eigenspace of J_X, respectively, such that we have:*

$$(TX)^{\mathbb{C}} = (TX)_{(1,0)} \oplus (TX)_{(0,1)}$$

Proof. The proof is given below:

(1) Given $p \in X$, choose a holomorphic coordinate chart (U, φ) on a neighborhood of p, and define $J_X : TX|_U \to TX|_U$ by
$$J_X = D\varphi^{-1} \circ J_{\mathbb{C}^n} \circ D\varphi.$$
Wherever two holomorphic charts (U, φ) and (V, ψ) overlap, the transition map $\psi \circ \varphi^{-1}$ is a holomorphic map between open subsets of $\mathbb{C}^n$, so its differential commutes with $J_{\mathbb{C}^n}$ as in Example 10.9. Therefore,
$$\begin{aligned} D\psi^{-1} \circ J_{\mathbb{C}^n} \circ D\psi &= D\psi^{-1} \circ J_{\mathbb{C}^n} \circ (D\psi \circ D\varphi^{-1}) \circ D\varphi \\ &= D\psi^{-1} \circ (D\psi \circ D\varphi^{-1}) \circ J_{\mathbb{C}^n} \circ D\varphi \\ &= D\varphi^{-1} \circ J_{\mathbb{C}^n} \circ D\varphi. \end{aligned}$$
So J_X is well-defined. The fact that it satisfies $J_X \circ J_X = -\mathrm{Id}$ follows from the corresponding fact for $J_{\mathbb{C}^n}$.

(2) (Sketch) This follows since we have the decomposition
$$(T_pX)^{\mathbb{C}} = (T_pX)_{(1,0)} \oplus (T_pX)_{(0,1)}$$
for each $p \in X$.

This completes the proof. □

11. Cotangent Bundle & Differential Forms

11.1. **Smooth Differential Forms.** We first recall the notion of differential forms on a smooth manifold.

Definition 11.1. Let X be a smooth manifold. A **differential k-form** is a smooth map $\sigma : X \to \Lambda^k T^*X$ such that $\pi \circ \sigma = \mathrm{Id}_X$, where π denotes the projection map from $\Lambda^k T^*X$ onto X and
$$\Lambda^k T^*X = \coprod_{p \in X} \Lambda^k(T_p^*X).$$
is the k-th exterior bundle.

Remark 11.2. *It can be checked that $\Lambda^k(T^*M)$ has the structure of a smooth manifold of dimension $n + \binom{n}{k}$. Moreover, it can be shown that $\Lambda^k T^*X$ has the structure of a smooth vector bundle over X.*

Remark 11.3. *It can be checked that the set of smooth differential k-forms is a $\mathbb{R}$-vector space. We denote the vector space of smooth differential k-forms by $\Omega^k(X)$.*

We can use the discussion in Section 4 to define the wedge product of two differential forms in a pointwise manner:
$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p, \quad p \in X$$
Thus, if $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(X)$, then $\omega \wedge \eta \in \Omega^{k+l}(X)$.

Remark 11.4. *A 0-form is just a continuous real-valued function. If f is a 0-form and η is a k-form, we interpret the wedge product $f \wedge \eta$ to mean the ordinary product $f\eta$.*

If we define
$$\Omega^*(X) = \bigoplus_{k=0}^{n} \Omega^k(X),$$

then the wedge product turns $\Omega^*(X)$ into an associative, anti-commutative graded algebra. In any smooth chart $(U,(x^i))$ on X, a smooth k-form ω can be written locally as

$$\omega = \sum_I \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where the coefficients ω_I are smooth functions defined on the coordinate domain. Lemma 3.13 implies

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}} \right) = \delta^I_J,$$

Thus, the component functions ω_I of the k-form ω are determined by

$$\omega_I = \omega \left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}} \right).$$

The great thing about differential forms is that we can pullback differential forms. Let X, Y be smooth manifolds. Given a smooth map $F : X \to Y$ and a differential form $\omega \in \Omega^k(Y)$, the pullback along F, denoted as $F^*\omega$, gives a differential form $\Omega^k(X)$. We can describe it by its action on tangent vectors:

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k))$$

where dF_p is the differential at p. The pullback satisfies some nice properties:

Proposition 11.5. *Let X, Y be smooth manifolds, and let $F : X \to Y$ be a smooth function. The pullback satisfies the following properties:*

(1) *$F \colon \Omega^k(Y) \to \Omega^k(X)$ is linear over $\mathbb{R}$ for each k.*
(2) *$F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$.*
(3) *In any smooth chart,*

$$F\left(\sum_I \omega_I \, dy^{i_1} \wedge \ldots \wedge dy^{i_k} \right) = \sum_I (\omega_I \circ F) \, d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F).$$

Proof. The proof is given below:

(1) This is clear.
(2) It suffices to prove the claim pointwise:

$$\begin{aligned}(F^*(\omega \wedge \eta))_p(v_1, \ldots, v_k, w_1, \cdots w_l) &= (\omega \wedge \eta)_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k), dF_p(w_1), \ldots, dF_p(w_l)), \\ &= \omega_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k))\eta_{F(p)}\eta(dF_p(w_1), \ldots, dF_p(w_l)) \\ &= F^*(\omega_p)(v_1, \ldots, v_k) \wedge F^*(\eta_p)(w_1, \cdots w_l)\end{aligned}$$

Hence,

$$(F^*(\omega \wedge \eta)) = F^*(\omega_p) \wedge F^*(\eta_p).$$

(3) This follows from (1) and (2) and the observation that if η is a 0-form (a function), then $F^*(\eta) = (\eta \circ F)$ and:

$$F^*(\omega \wedge \eta) = (\eta \circ F) F^*(\omega)$$

This completes the proof. $\square$

Example 11.6. Let $\omega = dx\wedge dy$ on $\mathbb{R}^2$. Thinking of the transformation to polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ as an expression for the identity map with respect to different coordinates on the domain and codomain, we obtain

$$\begin{aligned} dx \wedge dy &= d(r\cos(\theta)) \wedge d(r\sin(\theta)) \\ &= (\cos(\theta)\, dr - r\sin(\theta)\, d\theta) \wedge (\sin(\theta)\, dr + r\cos(\theta)\, d\theta) \\ &= r\, dr \wedge d\theta. \end{aligned}$$

Let's now recall the definition of the exterior derivative. For $0 \le k \le n$, the exterior derivative is an operator

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

In a smooth chart $(U, (x^i))$ on X, if a smooth k-form ω can be written locally as

$$\omega = \sum_I \omega_I\, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

then in local coordinates d is defined as follows:

$$d\omega = \sum_I \sum_j \frac{\partial \omega_I}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

Before we proceed, we give an equivalent but coordinate-free definition of the exterior derivative. Let's start with small k's to find out the invariant formula of $d\omega$. Let $U \subseteq X$ be an open subset of X. We have the following:

(1) Let $k = 0$. Then $\omega = f \in C^\infty(U)$, and we can regard df as a $C^\infty(U)$-linear map

$$df : \mathfrak{X}(U) \to C^\infty(U)$$

such that

$$df(X) = Xf.$$

(2) Let $k = 1$. We want to regard $d\omega$ as a $C^\infty(U)$-bilinear map

$$d\omega : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U).$$

We write $\omega = \sum_i \omega_i dx^i$, $X = \sum_k X^k \partial_k$, and $Y = \sum_l Y^l \partial_l$. Then

$$\begin{aligned} d\omega(X,Y) &= \sum_{i,j,k,l} (\partial_j \omega_i) dx^j \wedge dx^i (X^k \partial_k, Y^l \partial_l) \\ &= \sum_{i,j} (\partial_j \omega_i) X^j Y^i - (\partial_j \omega_i) X^i Y^j \\ &= \sum_{i,j} X^j \partial_j(\omega_i Y^i) - \omega_i X^j \partial_j(Y^i) - Y^j \partial_j(\omega_i X^i) + \omega_i Y^j \partial_j(X^i) \\ &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]). \end{aligned}$$

So we arrive at

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

(3) Let $k = 2$. We want to regard $d\omega$ as a $C^\infty(U)$-bilinear map

$$d\omega : \mathfrak{X}(U) \times \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U).$$

By a tedious but similar computation as above, one can show that

$$d\omega(X,Y,Z) = X(\omega(Y,Z)) - Y(\omega(X,Z)) + Z(\omega(X,Y)) - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X).$$

So we are naturally led to the following the invariant formula for d:

Proposition 11.7. *Let X be a smooth n-manifold. Let $0 \leq k \leq n$ and let $U \subseteq X$ be an open set of X. For any $\omega \in \Omega^k(U)$, the $(k+1)$-form $d\omega$, viewed as a $C^\infty(U)$-multilinear map*

$$d\omega : \underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{(k+1)-\text{times}} \to C^\infty(U),$$

is given by the formula

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) := &\sum_i (-1)^{i-1} X_i \left(\omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1})\right) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}) \end{aligned}$$

Proof. The proof is skipped. ☐

We end with some properties of the exterior derivative:

Proposition 11.8. *Let X be a smooth n-manifold. Let $0 \leq k, l \leq n$ and let $U \subseteq X$ be an open set of X. Suppose $\omega \in \Omega^k(U)$, $\eta \in \Omega^l(U)$, $X \in \mathfrak{X}(U)$.*

(1) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
(2) $d \circ d = 0$.
(3) *Let Y be a smooth m-manifold and let $V \subseteq Y$ be an open set. Let $F : U \to V$ be a smooth map. Then*

$$d \circ F^* = F^* \circ d$$

Proof. The proof is given below:

(1) Since d is linear, it is enough to assume

$$\omega = f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad \eta = g\, dx^{j_1} \wedge \cdots \wedge dx^{j_l}$$

with indices set $I \cap J = \emptyset$. Then the formula follows from a direct computation:

$$\begin{aligned} d(\omega \wedge \eta) &= d(fg\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}) \\ &= \sum_i \partial_i (fg)\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &= \sum_i (\partial_i f)\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \eta + (-1)^k \omega \wedge \sum_i (\partial_i g)\, dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

(2) We first check this for $k = 0$:

$$\begin{aligned} d(df)(X, Y) &= X(df(Y)) - Y(df(X)) - df([X, Y]) \\ &= X(Y(f)) - Y(X(f)) - [X, Y]f = 0 \end{aligned}$$

For $k > 0$, by linearity we may assume

$$\omega = f\, dx^1 \wedge \cdots \wedge dx^k$$

Since $ddf = 0$ and $ddx^i = 0$, we get

$$\begin{aligned} d(d\omega) &= d(df \wedge dx^1 \wedge \cdots \wedge dx^k) \\ &= d(df) \wedge dx^1 \wedge \cdots \wedge dx^k + \sum_i (-1)^i df \wedge dx^1 \wedge \cdots \wedge d(dx^i) \wedge \cdots \wedge dx^k = 0 \end{aligned}$$

(3) For $0 \le k \le m$, let $\omega \in \Omega^k * (V)$. For $k = 0$, $\omega = f \in C^\infty(V)$ and

$$(\varphi^* df)_p(X_p) = df_{F(p)}(dF_p(X_p)) = d(F^* f)_p(X_p).$$

In general, assume

$$\omega = f\, dx^1 \wedge \cdots \wedge dx^k$$

By Proposition 11.5, we have

$$\begin{aligned} F^*(d\omega) &= F^*(df \wedge dx^1 \wedge \cdots \wedge dx^k) \\ &= F^*(df) \wedge F^*(dx^1) \wedge \cdots \wedge F^*(dx^k) \\ &= d(F^* f) \wedge d(F^* x^1) \wedge \cdots \wedge d(F^* x^k) \\ &= d(F^* f\, d(F^* x^1) \wedge \cdots \wedge d(F^* x^k)) \\ &= d(F^* \omega). \end{aligned}$$

This completes the proof. □

11.2. **Cotangent Bundle.** Let X be a complex n-manifold and let $\pi \colon T^*X \to X$ be the smooth rank-$2n$ cotangent bundle over the underlying smooth manifold structure on X. We first discuss the complexification of T^*X. The details are similar to the complexification of the underlying smooth tangent bundle, but we repeat the details anyway. Define the complexification of T^*X to be the set

$$(T^*X)^{\mathbb{C}} = \coprod_{p \in X} (T_p^* X)^{\mathbb{C}}$$

together with the obvious projection $\pi_{\mathbb{C}} \colon (T^*X)^{\mathbb{C}} \to X$. For each smooth local trivialization $\Phi \colon \pi^{-1}(U) \to U \times \mathbb{R}^{2n}$, we define a local trivialization $\Phi^{\mathbb{C}} \colon (\pi^{-1})^{\mathbb{C}}(U) \to U \times \mathbb{C}^{2n}$ by

$$\Phi^{\mathbb{C}}(\xi) = \big(\pi^{\mathbb{C}}(\xi), (\Phi | T^* X_{\pi^{\mathbb{C}}(\xi)})^{\mathbb{C}}(\xi)\big).$$

Wherever two such trivializations (U, Φ) and (V, Ψ) overlap, we can write

$$\Psi \circ \Phi^{-1}(p, v) = (p, \tau(p)v)$$

for some smooth transition function $\tau \colon U \cap V \to \mathrm{GL}(2n, \mathbb{R})$. Clearly the transition function from $\Phi_{\mathbb{C}}$ to $\Psi_{\mathbb{C}}$ is the same:

$$\Psi^{\mathbb{C}} \circ (\Phi^{-1})^{\mathbb{C}}(p, v) = (p, \tau(p)v),$$

where now τ is considered as a map into $\mathrm{GL}(2n, \mathbb{C})$. It follows from the vector bundle chart lemma that $\pi^{\mathbb{C}} \colon (T^*X)^{\mathbb{C}} \to M$ has a unique structure as a smooth rank-$2n$ holomorphic vector bundle, with the maps constructed above as smooth local trivializations.

Definition 11.9. Let X be a complex manifold. A **holomorphic 1-form** is a section of $(T^*X)^{\mathbb{C}}$.

Example 11.10. Let $X = \mathbb{C}^n$. With (x^j, y^j) as smooth global coordinates for $\mathbb{C}^n$, the smooth global coframe $\{dx^j, dy^j\}$ forms a coframe for $T^*\mathbb{C}^n$, and also for $(T^*\mathbb{C}^n)^{\mathbb{C}}$. Note that we have

$$\begin{aligned}(T^*\mathbb{C}^n)^{\mathbb{C}} &= \mathrm{Span}_{\mathbb{C}}\langle dx^1, \dots, dx^n, dy^1, \dots, dy^n\rangle \\ &= \mathrm{Span}_{\mathbb{C}}\langle dz^1, \dots, dz^n, d\overline{z}^1, \dots, d\overline{z}^n\rangle.\end{aligned}$$

Here we have defined

$$dz^j = dx^j + i\, dy^j, \qquad d\overline{z}^j = dx^j - i\, dy^j$$

If $f \colon U \to \mathbb{C}$ is a smooth function on an open subset $U \subseteq \mathbb{C}^n$, we can write

$$\begin{aligned}df &= \frac{\partial f}{\partial x^j} dx^j + \frac{\partial f}{\partial y^j} dy^j \\ &= \frac{\partial f}{\partial x^j}\left(\frac{dz^j + d\overline{z}^j}{2}\right) + \frac{\partial f}{\partial y^j}\left(\frac{dz^j - d\overline{z}^j}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x^j} - i\frac{\partial f}{\partial y^j}\right) dz^j + \frac{1}{2}\left(\frac{\partial f}{\partial x^j} + i\frac{\partial f}{\partial y^j}\right) d\overline{z}^j \\ &= \frac{\partial f}{\partial z^j} dz^j + \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j\end{aligned}$$

11.3. **Holomorphic differential Forms.** Let X be a complex n-manifold. We now discuss holomorphic k-forms for $0 \leq k \leq n$. For $0 \leq k \leq n$, let $(\Lambda^k T^*X)^{\mathbb{C}}$ be the complexification of $\Lambda^k T^*X$. As a set, we have

$$(\Lambda^k T^*X)^{\mathbb{C}} = \coprod_{p \in X} (\Lambda^k(T_p^*X))^{\mathbb{C}}$$

A holomorophic vector bundle strucuture on $(\Lambda^k T^*X)^{\mathbb{C}}$ can be constructed in much the same way as that of the complex tangent bundle and complex cotangent bundle, so we don't provide additional details. This allows us to define holomorphic k-forms:

Definition 11.11. Let X be a complex manifold. A **holomorphic k-form** is a section of $(\Lambda^k X)^{\mathbb{C}}$ for $0 \leq k \leq n$

Remark 11.12. *If $k = 0$, a holomorphic 0-form is just a holomorphic function on X. If $k = 1$, then note that $\Lambda^1 T^*X = T^*X$ is the cotangent bundle and $(\Lambda^1 T^*X)^{\mathbb{C}}$ is the complexified cotangent bundle $(T^*X)^{\mathbb{C}}$. A holomorphic 1-form was defined in the previous section.*

Let's discuss the decomposition of $(\Lambda^k X)^{\mathbb{C}}$. Using results from Section 4 we have that

$$\begin{aligned}(\Lambda^k(T_p^*X))^{\mathbb{C}} &= \Lambda^k(T_p^*X)^{\mathbb{C}} \\ &= \Lambda^k((T_p^*X)_{(1,0)} \oplus (T_p^*X)_{(0,1)}) \\ &= \bigoplus_{p+q=k} \Lambda^p((T_p^*X)_{(1,0)}) \otimes_{\mathbb{C}} \Lambda^q((T_p^*X)_{(0,1)}) := \bigoplus_{p+q=k} \Lambda^{p,q}(T_p^*X)\end{aligned}$$

As a result, we have

$$\begin{aligned}(\Lambda^k T^*X)^{\mathbb{C}} &= \coprod_{p\in X} (\Lambda^k(T_p^*X))^{\mathbb{C}} \\ &= \coprod_{p\in X} \bigoplus_{p+q=k} \Lambda^p((T_p^*X)_{(1,0)}) \otimes_{\mathbb{C}} \Lambda^q((T_p^*X)_{(0,1)}) \\ &:= \coprod_{p\in X} \bigoplus_{p+q=k} \Lambda^{p,q}(T_p^*X) := \bigoplus_{p+q=k} \Lambda^{p,q}(T^*X)\end{aligned}$$

Definition 11.13. Let X be a complex n-manifold. Let $0 \leq k \leq 2n$ and $0 \leq p, q \leq n$ such that $p + q = k$. A **holomorphic** (p, q)**-form** is a section σ of $\Lambda^{p,q}(T^*X)$.

If $U \subseteq X$ is an open set of X corresponding to a holomorphic atlas then it is then clear that the following collection of forms constitutes a smooth local frame for $\Lambda^{p,q}(T^*X)$:

$$\{dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\overline{z}^{l_1} \wedge \cdots \wedge d\overline{z}^{l_q} : p + q = k,\ j_1 < \cdots < j_p,\ l_1 < \cdots < l_q\}.$$

Hence, in every local holomorphic coordinate chart $(U, (z^1, \ldots, z^n))$, $\sigma \in \Lambda^{p,q}(T^*X)$ can be expressed as

$$\sigma|_U = \sum_{\substack{j_1<\cdots<j_p, \\ l_1<\cdots<l_q}} dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\overline{z}^{l_1} \wedge \cdots \wedge d\overline{z}^{l_q}$$

Remark 11.14. *We use the notation $\mathscr{E}^k(X)$ to denote the space of holomorphic sections of $(\Lambda^k T^*X)^{\mathbb{C}}$, and $\mathscr{E}^{p,q}(X)$ for the space of holomorphic sections of $\Lambda^{p,q}(T^*X)$.*

For each $0 \leq p, q \leq n$, we have projection operators

$$\pi^{p,q} \colon (\Lambda^k T^*X)^{\mathbb{C}} \to \Lambda^{p,q}(T^*X)$$

Using the definition and properties of the wedge product, $\wedge$, and the exterior derivative, d, from the smooth manifold case, we have the following proposition:

Proposition 11.15. *Let X be a complex n-manifold.*

(1) *Let $\alpha \in \mathscr{E}^{p,q}(X)$. Then*
$$d\big(\mathscr{E}^{p,q}(X)\big) \subseteq \mathscr{E}^{p+1,q}(X) \oplus \mathscr{E}^{p,q+1}(X).$$
(2) *For each $0 \leq p, q \leq n$, there exists Dolbeault operators*
$$\partial \colon \mathscr{E}^{p,q}(X) \to \mathscr{E}^{p+1,q}(X). \qquad \bar{\partial} \colon \mathscr{E}^{p,q}(X) \to \mathscr{E}^{p,q+1}(X)$$
such that
$$\partial = \pi^{p+1,q} \circ d, \quad \bar{\partial} = \pi^{p,q+1} \circ d.$$
(3) *If $\alpha \in \mathscr{E}^{p,q}(X)$ and $\beta \in \mathscr{E}^{p',q'}(X)$*
$$\overline{\alpha} \in \mathscr{E}^{q,p}(M).$$
$$\alpha \wedge \beta \in \mathscr{E}^{p+p',q+q'}(M).$$
(4) *If $\alpha \in \mathscr{E}^k(X)$ and $\beta \in \mathscr{E}^l(X)$, then*
$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta,$$
$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^k \alpha \wedge \overline{\partial}\beta.$$

Proof. It suffices to work in local holomorphic coordinates.

(1) Choose holomorphic local coordinates $(z^1, \dots, z^n)$ and write

$$\alpha = \sum\nolimits_{J,L} \alpha_{J,L}\, dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q},$$

where $J = (j_1, \dots, j_p)$ and $L = (l_1, \dots, l_q)$ are strictly increasing multi-indices. We have

$$d\alpha = \sum\nolimits_{J,L} \sum_r \left(\frac{\partial \alpha_{J,L}}{\partial z^r} dz^r + \frac{\partial \alpha_{J,L}}{\partial \bar{z}^r} d\bar{z}^r \right) \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q},$$

The claim follows.

(2) This follows from (1).
(3) This is clear.
(4) This follows from (1), (2) and (3).

This completes the proof. □

11.4. **Dolbeault Cohomology.** We now discuss Dolbeault cohomology. Dolbeault cohomology is a fundamental tool in complex geometry, used to study the structure of complex manifolds. We start off with a basic prototype of holomorphic differential forms that will allow us to define the Dolbeault cohomology.

Lemma 11.16. *Let X, Y be a complex manifolds and let $F : X \to Y$ be a holomorphic map.*

(1) *If $\alpha \in \mathscr{E}^k(X)$, then we have*

$$\begin{aligned} d\alpha &= \partial\alpha + \overline{\partial}\alpha \\ \overline{\partial\alpha} &= \overline{\partial}(\overline{\alpha}), \\ \partial \circ \partial\alpha &= \overline{\partial} \circ \overline{\partial}\alpha = 0, \\ \partial \circ \overline{\partial}\alpha &= -\overline{\partial} \circ \partial\alpha. \end{aligned}$$

(2) *If $\alpha \in \mathscr{E}^{p,q}(Y)$, then we have*

$$\begin{aligned} F^*(\mathscr{E}^{p,q}(Y)) &\subseteq \mathscr{E}^{p,q}(X), \\ F^*(\partial\alpha) &= \partial(F^*\alpha), \\ F^*(\overline{\partial}\alpha) &= \overline{\partial}(F^*\alpha). \end{aligned}$$

Proof. The proof is given below:

(1) WLOG, let α be a (p, q) form. The first identity follows from (1) and (2) in Proposition 11.15. The second identity follows from the definition of conjugation. Note that

$$0 = d(d\alpha) = (\partial + \overline{\partial})(\partial + \overline{\partial})\alpha = \partial \circ \partial\alpha + (\partial \circ \overline{\partial}\alpha + \overline{\partial} \circ \partial\alpha) + \overline{\partial} \circ \overline{\partial}\alpha.$$

On the right-hand side, the first term is in $\mathscr{E}^{p+2,q}(X)$, the term in parentheses is in $\mathscr{E}^{p+1,q+1}(X)$, and the last term is in $\mathscr{E}^{p,q+2}(X)$. Since these spaces intersect only in the zero form, each of those three terms must be zero. Hence, the third and fourth identities follow.

(2) Choose holomorphic local coordinates $(z^1, \dots, z^n)$ and $(w^1, \dots, w^m)$. Note that we have

$$F^* dw^j = \frac{\partial F^j}{\partial z^l} dz^l,$$
$$F^* d\overline{w}^j = \frac{\partial \overline{F}^j}{\partial \overline{z}^l} d\overline{z}^l.$$

The first identity follows by a simple linearity argument. The second and third identities follow from the fact that F^* commutes with $\pi^{p+1,q}$, $\pi^{p,q+1}$ and d.

This completes the proof. $\square$

Lemma 11.16 allows us to define a set of biholomorphic invariants. Because $\overline{\partial} \circ \overline{\partial} = 0$, for each p we obtain a cochain complex known as the p-th Dolbeault complex:

$$0 \to \mathscr{E}^{p,0}(M) \xrightarrow{\overline{\partial}} \mathscr{E}^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{E}^{p,n}(M) \to 0.$$

Definition 11.17. Let X be a complex n-manifold. Let $0 \le p, q \le n$. The **(p,q)-Dolbeault cohomology group** is defined as

$$H^{p,q}(X) = \frac{\mathrm{Ker}(\overline{\partial} : \mathscr{E}^{p,q}(M) \to \mathscr{E}^{p,q+1}(M))}{\mathrm{Im}(\overline{\partial} : \mathscr{E}^{p,q-1}(M) \to \mathscr{E}^{p,q}(M))}$$

It follows from Lemma 11.16 that the construction of Dolbeault cohomology is functorial and that it is indeed a biholomorphic invariant.

Definition 11.18. Let X be a complex n-manifold and and let $0 \le p, q \le n$. The **Hodge numbers** of X are given by

$$h^{p,q}(X) = \dim H^{p,q}(X)$$

The Dolbeault cohomology groups measure the the extent to which a 'which $\overline{\partial}$-closed forms fail to be $\overline{\partial}$-exact.' The $\overline{\partial}$ Poincaré lemma states that a $\overline{\partial}$-closed form can always be locally written as a $\overline{\partial}$-exact form.

Proposition 11.19. ***($\overline{\partial}$-Poincaré Lemma)*** *Let X be a complex n-manifold and let $0 \le p, q \le n$. If $\omega \in \Omega^{p,q}(X)$ is a smooth form that satisfies $\overline{\partial}\omega = 0$ for $q \ge 1$, then in a neighborhood of each point there is a $\eta \in \Omega^{p,q-1}(X)$ that is a smooth form such that $\overline{\partial}\eta = \omega$.*

Remark 11.20. *Nore that Proposition 11.19 is about smooth (p,q) forms. This is because the proof works with functions*

Proof. $\square$