BANACH & C^*-ALGEBRAS

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Abstract. This document comprises notes on Banach algebras and C^*-algebras. A portion of these notes was taken during my participation in the Groundwork for Operator Algebras Lecture Series (GOALS) workshop at IPAM, UCLA. If you come across any typos, please send corrections to junaid.aftab1994@gmail.com.

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Remark 0.1. *Unless otherwise specified, we work over* $\mathbb{K} = \mathbb{C}$*, the field of complex numbers in these notes.*

Part 1. Banach Algebras

1. Why Banach Algebras?

Banach algebras are perhaps the most general type of operator algebra. They also provide a natural setting for studying classical topics in functional analysis, such as spectral theory. Key concepts like the spectrum, resolvent sets, and spectral radius can be examined within this framework, facilitating the analysis of operator behavior on Hilbert spaces and beyond. Additionally, C^*-algebras and von Neumann algebras form special classes of Banach algebras. Therefore, studying Banach algebras is a fundamental step in the broader study of operator algebras.

2. Definitions & Examples

We begin with a detailed discussion of definitions and examples in Banach algebra theory. If $\mathscr{H}$ is a Hilbert space, note that the Banach space, $\mathscr{B}(\mathscr{H})$, of bounded linear operators on $\mathscr{H}$ has more structure than that of a Banach space. Indeed, if $T, S \in \mathscr{B}(\mathscr{H})$, then $T \circ S \in \mathscr{B}(\mathscr{H})$ such that

$$\|T \circ S\| \leq \|T\|\|S\|$$

This observation motivates the definition of a Banach algebra. We will proceed in multiple steps.

Definition 2.1. Let V be a complex vector space.

- V is an **algebra** if the underlying abelian group admits a multiplication operation

$$\cdot : V \times V \to V \qquad (x, y) \mapsto x \cdot y$$

  endowing V with a ring structure compatible with the given scalar multiplication.
- V is a **normed algebra** if it is equipped with a submultiplicative norm.

Remark 2.2. *If V is a (normed) algebra, we say V is a unital (normed) algebra if it admits an identity element. That is, there exists a $e \in V$ such that $e \cdot x = x \cdot e = x$ for all $x \in V$. It is a simple exercise to check that the identity in a unital algebra is unique. The proof is identical from group theory.*

Remark 2.3. *If is V a (normed) algebra, we say V is an abelian (normed) algebra if $x \cdot y = y \cdot x$ for all $x, y \in V$*

Remark 2.4. *From now on, we abbreviate the ring multiplication operator $x \cdot y$ as simply xy for x, y in an algebra.*

If V is a normed algebra, then the norm induces a metric on V which in turn induces a topology on V called the norm topology. Here is a sample proposition:

Lemma 2.5. *Let V be a normed algebra. Addition, scalar multiplication and multiplication are are continuous in the norm topology on V.*

Proof. Let's consider the multiplication operation. Let $x_n \to x$ and $y_n \to y$. The submultiplicativity of the norm implies that

$$\|xy - x_n y_n\| \leq \|xy - xy_n\| + \|xy_n - x_n y_n\| \leq \|x\|\|y - y_n\| + \|x - x_n\|\|y_n\| \to 0.$$

It is clear that addition and scalar multiplication are continuous. □

Definition 2.6. A **Banach algebra**, A, is a normed algebra that is complete in the metric topology induced by the norm.

In other words, a Banach algebra, A, is a Banach space endowed with a sub-multiplicative operation making X into a normed algebra.

Remark 2.7. *If A is unital Banach algebra with identity element $e \in A$, then note that we have*

$$\|e\| = \|ee\| \leq \|e\|\|e\|$$

Hence, $\|e\| \geq 1$. In a C^*-algebra, we have $\|e\| = 1$. Observe that for any $r \in \mathbb{R}$ with $r \geq 1$, $(A, r\|\cdot\|)$ remains a Banach algebra. Moreover, note that $\|\cdot\|$ and $r\|\cdot\|$ are equivalent norms. Therefore, it is possible to modify the norm of any Banach algebra such that the unit has a norm of 1 with respect to this new norm.

Notice that a subalgebra is itself an algebra. A subalgebra of a normed algebra is a normed algebra. The closure of a subalgebra of a normed algebra is a normed algebra. Therefore the closure of any subalgebra of a Banach algebra is again a Banach algebra. This observation generates a long list of examples:

Example 2.8. The following is a list of some basic examples of Banach algebras:

(1) Let S be a set, and let $\ell^\infty(S)$ be the collection of all bounded complex-valued functions on S. Then $\ell^\infty(S)$ is a Banach algebra with respect to the usual pointwise operations defined as follows:

$$\begin{aligned} (f+g)(s) &:= f(s) + g(s), \\ (fg)(s) &:= f(s)g(s), \\ (\lambda f)(s) &:= \lambda f(s) \end{aligned}$$

for all $s \in S$. It is a commutative unital Banach algebra with identity given by the constant function $x \mapsto 1$. The norm is given by

$$\|f\|_\infty = \sup_{s \in S} |f(s)|.$$

(2) Let X be a locally compact Hausdorff space, X. Let $C_b(X)$ denote the space of bounded continuous complex-valued functions on X. It can be checked that $C_b(X)$ is a closed subalgebra of $\ell^\infty(X)$. Hence, $C_b(X)$ is a unital commutative Banach algebra.

(3) Let X be a locally compact Hausdorff space, X. Let $C_0(X)$ be the the space of continuous complex-valued functions on X that vanish at infinity. It can be checked that $C_0(X)$ is a closed subalgebra of $C_b(X)$. Hence, $C_0(X)$ is a commutative Banach algebra. It is unital if and only if X is compact.

(4) Let $\mathscr{H}$ be a Hilbert space. Then $\mathscr{B}(\mathscr{H})$ is a (generally non-commutative) unital Banach algebra with multiplication operation given by composition, identity given by the identity operator, and norm given by the operator norm. In particular, if $\mathscr{H} = \mathbb{C}^n$ then $M_n(\mathbb{C})$ is a Banach algebra under matrix multiplication and the operator norm.

Remark 2.9. *If X is locally compact and Hausdorff, then $C_b(X)$ contains many functions due to Urysohn's Lemma. If X is locally compact but not compact, then $C_b(X)$ is likely non-separable. Let $X = \mathbb{R}$. Consider the subset $K \subseteq C_b(\mathbb{R})$ consisting of functions that are either 0 or 1 at the integers. There is an uncountable subset S of K such that:*

$$\|f - g\| \geq 1, \quad \textit{whenever } f, g \in S \textit{ with } f \neq g.$$

Given a countable subset B of $C_b(\mathbb{R})$, it follows that there is an $s \in S$ that is at least $\frac{1}{2}$ distance away from every element of B. Thus, B is not dense in $C_b(\mathbb{R})$.

Remark 2.10. *If X is a locally compact Hausdorff space, we have that $C(X)$ is also a Banach algebra. In general, we have the following inclusions of Banach algebras:*

$$C_0(X) \subsetneq C_b(X) \subsetneq C(X)$$

If X is a compact Hausdorff space, then

$$C_0(X) = C_b(X) = C(X)$$

Definition 2.11. Let A and B be Banach algebras. A morphism $\phi : A \to B$ is a continuous $\mathbb{C}$-linear, multiplicative map. Moreover, ϕ is an isometric isomorphism if it has a continuous inverse and ϕ is an isometry.

Remark 2.12. *If A, B are unital Banach algebras, then a morphism $\phi : A \to B$ is unital morphism if $\phi(e_A) = e_B$.*

Example 2.13. Let X be a compact topological space and let $Y \subseteq X$ be a compact subspace. Then the restriction of functions is a momorphism of Banach algebras from $C(X)$ to $C(Y)$. This includes the special case when $Y = \{x\}$, consisting of a single element. In this case, $C(Y) \cong \mathbb{C}$, and the restriction is the evaluation homomorphism $\delta_x : C(X) \to \mathbb{C}$ mapping f to $f(x)$.

Commutative Banach algebras are much easier to study. In spite of the fact that we are most interested in algebras modeling $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ — which are not commutative — the theory of commutative Banach algebras plays a very important role in the sequel. We discuss an important example of a commutative Banach algebras to end this section:

Example 2.14. Consider $L^1(\mathbb{R})$, the Banach space of Lebesgue integrable complex-valued functions on $\mathbb{R}$. The convolution of two functions f and g in $L^1(\mathbb{R})$ is defined as

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\, dy.$$

$L^1(\mathbb{R})$ is a commutative Banach algebra. It is easy to check that the $+$ and $*$ operations on $L^1(\mathbb{R})$ satisfy the axioms of an abelian algebra. That is, we have,

$$\begin{aligned}
(f * g) * h &= f * (g * h) \\
f * (g + h) &= f * g + f * h \\
(f + g) * h &= f * h + g * h \\
f * g &= g * f
\end{aligned}$$

Moreover, $*$ is sub-multiplicative. Indeed, we have,

$$\begin{aligned}
\|f * g\|_1 &= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)||g(y)|\; dy\; dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)||g(y)|\; dx\; dy \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)|\; dx \right] |g(y)|\; dy \\
&= \|f\|_1 \int_{\mathbb{R}} |g(y)|\; dy = \|f\|_1\, \|g\|_1
\end{aligned}$$

$L^1(\mathbb{R})$ is a non-unital Banach algebra. Assume that there is a $f \in L^1(\mathbb{R})$ is a unit. For each $g \in L^1(\mathbb{R})$, we have by the convolution theorem,

$$\hat{g} = \widehat{f * g} = \hat{f}\hat{g}$$

But if $g \in L^1(\mathbb{R})$ is a Gaussian, then $\hat{g}$ doesn't vanish. Hence, $\hat{f} \equiv 1$. But the Riemann Lebesgue lemma implies that $\hat{f} \in C_0(\mathbb{R})$. This is a contradiction.

Remark 2.15. *The convolution on $L^1(\mathbb{R})$ is first defined on the dense subspace $C_c(\mathbb{R})$, and subsequently be extended by continuity to $L^1(\mathbb{R})$.*

Example 2.16. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the open unit disk. Let $\mathscr{O}(\mathbb{D}) \subseteq C(\mathbb{D})$ be the subalgebra of holomorphic functions on $\mathbb{D}^\circ$. By Morera's Theorem, the uniform limit of holomorphic functions is holomorphic, which implies that $\mathscr{O}(\mathbb{D})$ is a closed subalgebra of $C(\mathbb{D})$. Let $\mathscr{A}(\mathbb{D})$ be the image of $\mathscr{O}(\mathbb{D})$ in $C(\mathbb{S}^1)$ via the restriction map. Since the map $g \mapsto g|_{\mathbb{S}^1}$ is isometric by the Maximum Modulus Principle, $\mathscr{A}(\mathbb{D})$ is complete and therefore closed in $C(\mathbb{S}^1)$. Hence, $\mathscr{A}(\mathbb{D})$ is a closed subalgebra of $C(\mathbb{S}^1)$ and forms a Banach algebra, commonly referred to as the disk algebra. It consists of those functions in $C(\mathbb{S}^1)$ that have holomorphic extensions to $\mathbb{D}$.

3. Unitization

When a Banach algebra, A, does not contain a unit, we can always add one, as follows. Form the vector space

$$A' := A \oplus \mathbb{C},$$

and make this into an algebra by means of

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$$

for each $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. In other words, $(0, 1) \in A'$, which can be identified with $1 \in \mathbb{C}$, is the identity, e', in A'. Furthermore, we can define a norm on A' by

$$\|(a, \lambda)\|_{A'} := \|a\|_A + |\lambda|$$

In particular, $\|e'\| = |1| = 1$. We have,

$$\begin{aligned} \|(a, \lambda)(b, \mu)\|_{A'} &\leq \|a\|_A\|b\|_A + |\lambda|\|b\|_A + |\mu|\|a\|_A + |\lambda||\mu| \\ &= \|(a, \lambda)\|_{A'}\|(b, \mu)\|_{A'} \end{aligned}$$

Hence, A' is a unital Banach algebra containing A.

Remark 3.1. *If A already has a unit e, then the algebra A' is isomorphic to the direct sum $A \oplus \mathbb{C}$ of the algebras A and $\mathbb{C}$, where we define multiplication component-wise. The isomorphism from A' to $A \oplus \mathbb{C}$ is given by*

$$(a, \lambda) \mapsto (a + \lambda e) \oplus \lambda$$

As a crucial example, we compute the unitization $C_0(X)'$. We recall the one-point compactification X_∞ of the space X. Let ∞ denote a new point, and define $X_\infty = X \cup \{\infty\}$, where X_∞ is X with an additional point. A set $U \subseteq X_\infty$ is open if it is either open in X or contains ∞ and the set $X \setminus U$ is compact in X. Every continuous function in $C(X_\infty)$ restricts to a continuous function on X, so $C_0(X)$ can be identified with the subspace of continuous functions on X_∞ that vanish at ∞.

Proposition 3.2. *Let X be a locally compact non-compact Hausdorff space. There is a topological (non-isometric) isomorphism of Banach algebras $C(X_\infty) \cong C_0(X)'$.*

Proof. Extending every $f \in C_0(X)$ by zero to X_∞, we consider $C_0(X)$ as a subspace of $C(X_\infty)$. Define $\psi : C_0(X)' \to C(X_\infty)$ by $\psi(f,\lambda) = f + \lambda 1_X$, where $1(x) = 1$ for all $x \in X_\infty$. Moreover, define $\phi : C(X_\infty) \to C_0(X)'$ by $\phi(f) = (f|_{C(X)}, 0)$. Then ψ is an isomorphism of algebras with inverse ϕ as can be easily checked. For the norms, we have

$$\|\psi(f,\lambda)\|_{C(X_\infty)} = \sup_{x\in X_\infty} |f(x) + \lambda| \leq \sup_{x\in X} |f(x)| + |\lambda| = \|(f,\lambda)\|_{C(X)'}.$$

This shows that ψ is continuous. It can also be checked that ϕ is continuous. This proves the claim. □

4. Ideals

We can define the notion of an ideal in a Banach algebra.

Definition 4.1. Let A be a Banach algebra. A subspace $I \subseteq A$ of a commutative Banach algebra A is called a left ideal if for any $a \in I$, it follows that $ab \in I$ for all $b \in A$. A right ideal is defined similarly.

Proposition 4.2. *Let A be a Banach algebra and let $I \subseteq A$ be a closed two-sided left proper ideal. The quotient space A/I is a Banach algebra with respect to the quotient norm.*

Proof. It is a standard fact from Banach space theory that A/I is Banach space with the quotient norm. We only show that the quotient norm is sub-multiplicative. For given $a, b \in A$ and for every $\epsilon > 0$, by the definition of the quotient norm, there exist $m, n \in I$ such that

$$\|a + m\| \leq \|\pi(a)\| + \epsilon, \qquad \|b + n\| \leq \|\pi(b)\| + \epsilon$$

Since $(a + m)(b + n) \in ab + I$, we have

$$\begin{aligned}\|\pi(a)\pi(b)\| &= \|\pi((a + m)(b + n))\| \\ &\leq \|(a + m)(b + n)\| \\ &\leq \|a + m\|\|b + n\| \\ &\leq \|\pi(a)\|\|\pi(b)\| + \epsilon(\|\pi(a)\| + \|\pi(b)\| + \epsilon).\end{aligned}$$

This holds for every ϵ, and so it implies the desired claim. □

Example 4.3. Let A be a Banach algebra.

(1) There are two trivial ideals, the one consisting of the zero element and the one consisting of A itself.
(2) Any ideal, I, that contains the unit element e is equal to A.
(3) If A is a non-unital algebra, is clear that the embedding of A into A' is linear and isometric and that A sits inside of A' as a closed ideal[1]. Hence, we can in a way think of A' as the smallest unital Banach algebra in which A sits as an ideal.

Definition 4.4. Let A be a Banach algebra. An ideal $I \subseteq A$ is a left (resp. right) maximal ideal if it is not contained in any other non-left (resp. right) trivial ideal.

Zorn's lemma implies that maximal ideals always exist.

Proposition 4.5. *Let A be a Banach algebra. Any non-trivial left (resp. right) ideal is a subset of a left (resp. right) maximal ideal.*

[1] In fact, as a maximal ideal, given that its co-dimension is equal to 1.

Proof. Denote the ideal by I. A partial order among left (resp. right) ideals containing I is established through inclusion. Consider any chain of such non-trivial left (resp. right) ideals I_α, i.e., for any $\alpha \neq \beta$, either $I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha$. We claim that $U = \bigcup_\alpha I_\alpha$ contains I, is a left (resp. right) ideal, and hence an upper bound. Clearly, U is a subspace since all I_β are subspaces. Any $x \in U$ is in some I_α, and if $y \in A$ is any element, we have that $xy \in I_\alpha$ and thus in U. Since e is not in any of the I_β, it is not in U, and hence U is non-trivial. That $I \subseteq U$ is evident. By Zorn's lemma, there exists a maximal element M, i.e., M is an ideal such that whenever V is an ideal that contains I and M, then $V = M$. Hence, M is a maximal ideal. □

Corollary 4.6. *Let A be a Banach algebra and let $I \subseteq A$ be a left (resp. right) ideal. Then $\overline{I}$ is a left (resp. right) ideal. In particular any maximal ideal is closed.*

Proof. This follows from the continuity of multiplication and the fact that the closure of a non-trivial ideal is non-trivial. □

We can extend the notion of a maximal ideal to a non-unital Banach algebra.

Definition 4.7. Let A be a non-unital Banach algebra. A left (resp. right) ideal $I \subseteq A$ is called regular if A/I contains a unit.

Remark 4.8. *A Zorn's lemma argument can be used to show that a non-unital Banach algebra contains a maximal regular ideal.*

5. Spectrum

In this section, we consider the notion of the spectrum of an element in a Banach algebra. The spectrum of an element in a Banach algebra generalizes the notion of eigenvalues of a matrix. For each element in a Banach algebra, we show that its spectrum is a non-empty, compact set. We then define the spectral radius of an element of a Banach algebra, and we prove the spectral radius formula.

Definition 5.1. Let A be a unital Banach algebra with unit e. An element $x \in A$ is called **invertible** if there exists $y \in A$ such that

$$xy = yx = e$$

The element y is called the inverse of x, denoted x^{-1}.

Remark 5.2. *It can be easily checked that the inverse of an element is unique. Let $\mathrm{GL}(A)$ denote the set of invertible elements of A. Then $\mathrm{GL}(A)$ forms a group, and $(xy)^{-1} = y^{-1}x^{-1}$ for $x, y \in \mathrm{GL}(A)$.*

Clearly, we need A to have a unit in order to define the notion of invertibility.

Definition 5.3. Let A be a unital Banach algebra with unit e. The **resolvent of a in A**, denoted as $\rho_A(a)$, is the set,

$$\rho_A(a) = \{z \in \mathbb{C} \mid a - ze \text{ such that } a - ze \text{ is invertible}\}$$

The **spectrum of a in A**, denoted as $\rho_A(a)$, is the set,

$$\sigma_A(a) = \{z \in \mathbb{C} \mid a - ze \text{ such that } a - ze \text{ is not invertible}\}$$

Remark 5.4. *Clearly, we have, $\sigma_A(a) = \rho_A(a)^c$ for each $a \in A$.*

When A has no unit, the resolvent and the spectrum are defined through the embedding of A in its unitization $A' = A \oplus \mathbb{C}$. Hence, we define $\sigma_A(a) = \sigma_{A'}(a)$ for each $a \in A$.

Lemma 5.5. *Let A be a non-unital Banach algebra and let A' be its unitization. Then*

$$\sigma_{A'}(a) = \sigma_A(a) \cup \{0\}$$

Proof. Let e' denote the identity in A'. Then $0 \in \sigma_A(x)$ because otherwise $e' = x^{-1}x \in A$, a contradiction. Thus, $\sigma_A(x) \cup \{0\} = \sigma_A(x) := \sigma_{A'}(x)$. □

Example 5.6. Here is a list of basic examples of the spectrum of some concrete Banach algebras.

(1) When $A = M_n(\mathbb{C})$ is the algebra of $n \times n$ matrices, the spectrum of $a \in A$ is just the set of eigenvalues of a.
(2) When $A = C(X)$ for some compact topological space,the spectrum of $f \in A$ is just the range of f. Indeed,

$$\begin{aligned}\sigma_A(f) &= \{z \in \mathbb{C} \mid f - z1 \text{ is not invertible }\}\\ &= \{z \in \mathbb{C} \mid f - z1 = 0 \text{ for some } x \in X\ \}\\ &= \{z \in \mathbb{C} \mid z = f(x) \text{ for some } x \in X\ \} = \text{Range}(f)\end{aligned}$$

(3) Let X be a locally compact, non-compact topological space and let $A = C_0(X)$. We have that $\text{Range}(f) \cup \{0\} \subseteq \sigma_{A'}(f)$. If $\lambda \neq 0$ such that $\lambda \in f(X)$, then consider

$$g(x) = \frac{f(x)}{\lambda(\lambda - f(x))}$$

$g(x)$ is continuous on X. Moreover, because $f \in C_(X)$ implies $g \in C_0(X)$. It is easily verified that

$$\left(\frac{1}{\lambda} + g(x)\right)(\lambda - f(x)) = 1$$

Hence,

$$\sigma_{A'}(f) = \text{Range}(f) \cup \{0\}$$

This is consistent with the result we would get if were to apply the characterization of the unitization of $C_0(X)$.

We can establish numerous algebraic properties of the spectrum. Here is a sample proposition:

Proposition 5.7. *Let A, B be unital Banach algebras.*

(1) *If $a, b \in A$, then*
$$\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}.$$
(2) *If $a, u \in A$, and u is invertible, then*
$$\sigma_A(uau^{-1}) = \sigma_A(a).$$
(3) *If $\phi : A \to B$ is a unital morphism, then*
$$\sigma_B(\phi(a)) \subseteq \sigma_A(a)$$

Proof. The proof is given below:

(1) Indeed, let $0 \neq \lambda \in \rho_A(ab)$ and set $u := (ab - \lambda)^{-1}$. Hence $abu = uab = 1 + \lambda u$, and from this we obtain

$$\begin{aligned}(ba - \lambda)(bua - 1) &= \lambda \\ (bua - 1)(ba - \lambda) &= \lambda.\end{aligned}$$

Thus $ba - \lambda$ is invertible, and so $\lambda \in \rho_A(ba)$.

(2) Note that we have

$$uau^{-1} - \lambda e = uau^{-1} - \lambda ueu^{-1} = u(a - \lambda e)u^{-1}$$

It is then clear that $uau^{-1} - \lambda e$ is invertible if and only if $u(a - \lambda e)u^{-1}$ is invertible. The claim follows.

(3) Let $\lambda \in \sigma_B(\phi(a))$. Then

$$\phi(a) - \lambda e_B = \phi(a - \lambda e_A)$$

is not invertible. Since ϕ preserves invertibility, $a - \lambda e_A$ is not invertible. Hence, $\lambda \in \sigma_A(a)$.

This completes the proof. □

Remark 5.8. *Invertibility sometimes depends on the algebra in which one allows the inverse to exist. For instance, if $A \subseteq B$ is a unital subalgebra of a unital Banach algebra B, then by Proposition 5.7(3), $\rho_A(a) \subseteq \rho_B(a)$, but the containment may be strict. For example, consider the Banach subalgebra $\mathscr{A}(\mathbb{D}) \subseteq C(\mathbb{T})$. Here $\mathscr{A}(\mathbb{D})$ is the disk algebra. The function $f(z) = z$ in $C(\mathbb{T})$ is invertible in $C(\mathbb{T})$ but not in $A(\mathbb{D})$, since its inverse would have to be $\frac{1}{z}$, which has a singularity at the origin.*

The rest of the section is devoted to proving some crucial properties of the spectrum of an element in a Banach algebra. We first prove some important properties about invertible elements in a Banach algebra.

Lemma 5.9. *Let A be a unital Banach algebra, and let $a \in A$.*

(1) *If $\|a\| < 1$, then $a - e$ is invertible with inverse $\sum_{n=0}^{\infty} a^n$.*
(2) *If $\|a - e\| < 1$, then $a - e$ is invertible with inverse $\sum_{n=0}^{\infty} (a - e)^n$*
(3) *For each $z \in \mathbb{C} \setminus \{0\}$, $(a - ze)^{-1}$ always exists when $|z| > \|a\|$.*
(4) *The group $\mathrm{GL}(A)$ is an open set.*
(5) *The inversion map $a \mapsto a^{-1}$ of $\mathrm{GL}(A)$ is a homeomorphism.*
(6) *If $a \in \mathrm{GL}(A)$, then*

$$\sigma_A(a^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma_A(a)\}$$

Proof. The proof is given below:

(1) We first show that the sum is a Cauchy sequence. Indeed, for $n > m$, we have

$$\left\| \sum_{k=0}^{n} a^k - \sum_{k=0}^{m} a^k \right\| = \left\| \sum_{k=m+1}^{n} a^k \right\| \leq \sum_{k=m+1}^{n} \|a^k\| \leq \sum_{k=m+1}^{n} \|a\|^k$$

The sum on the right converges to zero by the theory of the geometric series. Since A is complete, the sum $\sum_{k=0}^{\infty} a^k$ is a well-defined element of A. Now compute

$$\sum_{k=0}^{n} a^k (a - e) = \sum_{k=0}^{n} (a^k - a^{k+1}) = e - a^{n+1}.$$

Hence
$$\left\| e - \sum_{k=0}^{n} a^k(a-e) \right\| = \|a^{n+1}\| \leq \|a\|^{n+1}.$$
which goes to 0 for $n \to \infty$, as $\|a\| < 1$ by assumption. Thus
$$\left(\lim_{n\to\infty} \sum_{k=0}^{n} a^k(a-e) \right) = e.$$
By a similar argument,
$$(a-e)\left(\lim_{n\to\infty} \sum_{k=0}^{n} a^k \right) = e.$$
So that, by continuity of multiplication in a Banach algebra, one finally has
$$\sum_{k=0}^{\infty} a^k := \lim_{n\to\infty} \sum_{k=0}^{n} a^k = (a-e)^{-1}.$$

(2) This follows from (1).

(3) Note that
$$(a - ze) = z(z^{-1}a - e)$$
Since $|z| > \|a\|$, we have that $|z^{-1}a| < 1$. The claim follows from (1).

(4) Given $a \in \mathrm{GL}(A)$, let $b \in A$ for which $\|b\| < \|a^{-1}\|^{-1}$. Observe that we have,
$$\|a^{-1}b\| \leq \|a^{-1}\|\|b\| < 1$$
Hence
$$a + b = a(e + a^{-1}b)$$
has an inverse, namely
$$(e + a^{-1}b)^{-1}a^{-1}$$
which exists by (1). It follows that
$$\{y \in A \mid \|x - y\| < \|a^{-1}\|^{-1}\| \subseteq \mathrm{GL}(A).$$
Indeed, if $\|y - x\| \leq \|a^{-1}\|^{-1}$, then if $b = y - x$, then $y = x + b$ is invertible from above. This shows that $\mathrm{GL}(A)$ is open in A.

(5) Let $a \in \mathrm{GL}(A)$. Let b such that $\|a - b\| < \frac{1}{2}\|a^{-1}\|^{-1}$. Note that,
$$\|b^{-1}\| - \|a^{-1}\| \leq \|b^{-1} - a^{-1}\| = \|b^{-1}(a-b)a^{-1}\| \leq \frac{1}{2}\|b^{-1}\|,$$
Hence, $\|b^{-1}\| \leq 2\|a^{-1}\|$. If $\varepsilon \in (0,1)$, we can now choose b such that $\|a - b\| < \frac{\varepsilon}{2}\|a^{-1}\|^{-2}$. Since the estimate above continues to hold, we have
$$\|b^{-1} - a^{-1}\| \leq \|b^{-1}(a-b)a^{-1}\| \leq 2\|a^{-1}\|^2\|b - a\| \leq \varepsilon$$
This shows that the inversion map is continuous. Since the inversion map is its own inverse, the claim follows.

(6) If $\lambda \neq 0$, then note that
$$a - \lambda e = a\lambda(e\lambda^{-1} - a^{-1}) = -a\lambda(a^{-1} - e\lambda^{-1})$$
Hence, if a is invertible and $\lambda \neq 0$, then $a - \lambda e$ is invertible if and only if $a^{-1} - \lambda^{-1}e$ is invertible. The claim follows from this observation.

This completes the proof. □

Example 5.10. Let A be a Banach algebra. Here is a list of basic examples of the topological group $\mathrm{GL}(A)$:

(1) Let $A = M_n(\mathbb{C})$. Then the unit group $\mathrm{GL}(A)$ is the group of invertible matrices. The continuity of the determinant function in this case gives another proof that $A^\times$ is open.
(2) Let $A = C(X)$ for a compact Hausdorff space X. Then the unit group $\mathrm{GL}(A)$ consists of all $f \in C(X)$ with $f(x) \neq 0$ for every $x \in X$.

Proposition 5.11. *Let A be a unital Banach algebra, and let $a \in A$. The spectrum of a is a compact set contained in the unit ball of radius $\|a\|$ in $\mathbb{C}$.*

Proof. Lemma 5.9(2) implies that

$$\sigma_A(a) \subseteq \{z \in \mathbb{C} \mid |z| \leq |a|\}$$

We show that $\sigma_A(a)$ is compact by showing that $\sigma_A(a)$ is a closed set. Given $a \in A$, we now define a function $f : \mathbb{C} \to A$ by

$$f(z) := a - ze.$$

Clearly, f is a continuous function. Because $\mathrm{GL}(A)$ is open in A by Lemma 5.9(3), it follows that $f^{-1}(\mathrm{GL}(A))$ is open in $\mathbb{C}$. But

$$f^{-1}(\mathrm{GL}(A)) = \{z \in \mathbb{C} \mid \text{such that } a - ze \text{ is invertible}\} = \rho_A(a)$$

Hence, $\sigma_A(a) = \rho_A(a)^c$ is a closed set. This shows that $\sigma_A(a)$ is a compact set. □

Is the spectrum of an element non-empty? The answer is yes, and we now prove it. Since the spectrum is a generalization of the study of eigenvalues of a complex-valued matrix, it is expected that some complex analysis will be required to prove the claim.

Definition 5.12. Let A be a unital Banach algebra, and let $W \subseteq \mathbb{C}$. A function $f : W \to A$ is a **Banach-algebra valued holomorphic function**

$$\frac{\partial f}{\partial z}(z_0) := \lim_{z \in W, z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for each $z_0 \in W$.

Remark 5.13. *Major results from single-valued complex analysis continue to hold, such as Cauchy's integral formula or Liouville's theorem. However, we don't include these results here.*

Proposition 5.14. *Let A be a Banach algebra and let $a \in A$. Then $\sigma_A(a) \neq \emptyset$.*

Proof. WLOG, assume that $a \neq 0$ because if $a = 0$, then $\sigma_A(0) = \{0\}$. Consider the function,

$$\begin{aligned} f : \rho_A(a) &\to A \\ z &\mapsto (a - ze)^{-1} \end{aligned}$$

We show that f is a Banach space-valued holomorphic function. For $\lambda \neq \mu \in \rho_A(a) = \mathbb{C}$, we have

$$\begin{aligned} (\lambda a - e)^{-1} &= (\lambda a - e)^{-1}(\mu a - e)(\mu a - e)^{-1} \\ &= (\lambda a - e)^{-1}((\mu - \lambda)e + \lambda a - e)(\mu a - e)^{-1} \\ &= ((\mu - \lambda)(\lambda a - e)^{-1} + e)(\mu a - e)^{-1} = (\mu - \lambda)(\lambda a - e)^{-1}(\mu a - e)^{-1} + (\mu a - e)^{-1} \end{aligned}$$

Therefore, we have,

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -(\lambda a - e)^{-1}(\mu a - e)^{-1}$$

Therefore,

$$\lim_{\mu \to \lambda} = \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -(\lambda a - e)^{-2} = -f(\lambda)^2$$

This shows that f is a Banach space-valued holomorphic function. If $|\lambda| > \|a\|$, we have,

$$\begin{aligned} f(\lambda) &= (\lambda a - e)^{-1} \\ &= \lambda^{-1}\left(e - \frac{a}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{a}{\lambda}\right)^n = \frac{1}{\lambda}e + \frac{1}{\lambda^2}a + \cdots . \end{aligned}$$

For $r > \|a\|$, let Γ_r denote a contour that is a circle of radius r. Since

$$\frac{1}{2\pi i}\int_{\Gamma_r} \lambda^m d\lambda = \delta_{m,-1},$$

we have,

$$a^n = \frac{1}{2\pi i}\int_{\Gamma_r} \lambda^n f(\lambda) d\lambda$$

for each $n \geq 0$. If $\sigma_A(a) = \emptyset$, then $\rho_A(a) = \mathbb{C}$ and f is an entire function. Moreover, f is bounded. Indeed, if $|\lambda| > 2\|a\|$, then

$$\|f(\lambda)\| \leq \frac{1}{|\lambda|}\frac{1}{1 - \frac{\|a\|}{|\lambda|}} \leq \frac{1}{|\lambda| - \|a\|} \leq \frac{1}{\|a\|}$$

Hence, f is constant. But then we would have[2]

$$0 \leq \|e\| = \|a^0\| = \frac{1}{2\pi}\int_{\Gamma_r} \|f(\lambda)\| d\lambda \leq \frac{M_r}{2\pi}\int_{\Gamma_r} d\lambda = 0$$

where $M_r = \max_{\lambda \in \Gamma_r} \|f(\lambda)\|$. Hence, $e = 0$, a contradiction. Hence, $\sigma_A(a)$ is non-empty for each $a \neq 0$. □

Definition 5.15. Let A be a unital Banach algebra and let $a \in A$. The **spectral radius** of $a \in A$ is defined as

$$r(a) := \sup\{|z| \mid z \in \sigma_A(a)\}.$$

Example 5.16. Let X be a compact Hausdorff space and let $A = C(X)$. Then $r(f) = \|f\|_\infty$.

Remark 5.17. *Let $A = M_2(\mathbb{C})$. Consider the family of matrices $\{A_t \mid t \in \mathbb{R}^+\}$ such that*

$$A_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Note that $\sigma_A(A_t) = \{1\}$ and $r(A_t) = \{1\}$. A is a C^*-algebra, and we can compute the norm of A_t as,

$$\|A_t\|^2 = r(A_t^T A^T)$$

Note that,

$$A_t^T A_t = \begin{pmatrix} 1 & t \\ t & 1 + t^2 \end{pmatrix}$$

[2]Here we assume that a Banach space valued integral exists and is well-defined.

Note that $r(A_t^T A_t) = 1/2(2 + t^2 + t\sqrt{4 + t^2})$. We see that thin this case, $r(A_t) < \|A_t\|$ for $t > 0$. Hence, we have that the spectral radius can be less than the norm of an element.

We end with a non-example of a Banach algebra.

Example 5.18. Let $A = \mathbb{C}[x]$, the algebra of complex polynomials in one variable, x. Let $w \in \mathbb{C}$ and $p \in \mathbb{C}[x]$ be a non-constant polynomial. Then $p - w$ is a non-constant polynomial, and so has a zero z_w by the fundamental theorem of algebra. This means that $p(z_w) = w$. Hence, p is surjective. That is, $\sigma_A(p) = \mathbb{C}$. Hence, $A = \mathbb{C}[x]$ is not a Banach algebra since $\sigma_A(a)$ is a non-compact set.

Remark 5.19. *Similarly, $B = \mathbb{C}(x)$, the field field of quotients for $A = \mathbb{C}[x]$ is not a Banach algebra.*

6. Polynomial Functional Calculus

We discuss polynomial functional calculus.

Proposition 6.1. *Let A be a unital Banach algebra, and let $a \in A$.*

(1) ***(Polynomial Spectral Mapping Theorem)*** *For a polynomial $p(z)$ on $\mathbb{C}[z]$, define $p(\sigma_A(a))$ as $\{p(z) \mid z \in \sigma_A(a)\}$. Then*

$$p(\sigma_A(a)) = \sigma_A(p(a)).$$

(2) ***(Spectral Radius Formula)*** *We have,*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

Proof. The proof is given below:

(1) We may suppose that p is not constant. If $\mu \subseteq \mathbb{C}$, there are elements $\lambda_0, \ldots, \lambda_n \in \mathbb{C}$, where $\lambda_0 \neq 0$, such that

$$p(z) - \mu = \lambda_0(z - \lambda_1 e) \cdots (z - \lambda_n e),$$

This follows because $\mathbb{C}$ is algebraically closed. Therefore,

$$p(a) - \mu = \lambda_0(a - \lambda_1 e) \cdots (a - \lambda_n e).$$

Note that $p(a) - \mu$ is invertible if and only if all $a - \lambda_1, \ldots, a - \lambda_n$ are invertible. Therefore, we have

$$\begin{aligned} p(a) - \mu \text{ is invertible} &\iff \text{at least one of } a - \lambda_i e \text{ is not invertible} \\ &\iff \lambda_i \in \sigma_A(a) \text{ for some } i \\ &\iff \mu = p(\lambda_i) \text{ for some } \lambda_i \in \sigma_A(a) \end{aligned}$$

The last statement follows since $p(\lambda) = \mu$ for each $\lambda = \lambda_1, \cdots, \lambda_n$. The claim now follows.

(2) If $\lambda \in \sigma_A(a)$ and $n \in \mathbb{N}$, (1) implies that we have $\lambda^n \in \sigma_A(a^n)$. Therefore,

$$|\lambda^n| \leq r(a^n) \leq \|a^n\|$$

Thus $|\lambda| \leq \|a^n\|^{1/n}$. Taking supremum as λ ranges over $\sigma_A(a)$ yields

$$r(a) \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$$

Let M_r as in Proposition 5.11. Using the formula for a^n as in Proposition 5.11, we have $\|a^n\| \leq r^{n+1}M_r$. Thus,

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \lim_{n\to\infty} r^{\frac{n+1}{n}} M_r^{\frac{1}{n}} = r$$

for $r > \|a\|$. Since $|\lambda| > r(a)$ implies that $\lambda \in \rho_A(a)$, it follows that if $r, r' > r(a)$, then Γ_r and $\Gamma_{r'}$ are homotopic in $\rho_A(a)$. Thus, the formula for a^n in Proposition 5.11 holds for all $r > r(a)$. It follows that

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq r(a).$$

This completes the proof. □

Remark 6.2. *Consider the algebra homomorphism*

$$\pi : \mathbb{C}[z] \to A$$

such that $\pi(1) = e$ *and* $\pi(z) = a$ *for some* $a \in A$. *We then have,*

$$\pi\Big(\sum_{i=0}^{n} c_i z^i\Big) = \sum_{i=0}^{n} c_i a^i$$

This homomorphism is called the polynomial functional calculus for a. *Hence, Proposition 5.11(1) is a statement about the spectrum of the polynomial functional calculus for* a.

Remark 6.3. *There is also holomorphic functional calculus that is not discussed in the notes.*

Corollary 6.4. *(**Gelfand-Mazur**) Let* A *be a unital Banach algebra in which every non-zero element is invertible. Then* A *is isometrically isomorphic to* $\mathbb{C}$.

Proof. Let $a \in A$. Since $\sigma_A(a) \neq \emptyset$, there is some $\lambda_a \in \mathbb{C}$ such that $a - \lambda_a e = 0$ is not invertible. By assumption, $a - \lambda_a e$. Hence, $a = \lambda_a \cdot e$. Define the map,

$$h : A \to \mathbb{C} \qquad h(a) = \lambda_a$$

It is easy to check that h is a linear map. Moreover, we have,

$$ab - \lambda_a\lambda_b \cdot e = ab - \lambda_a \cdot b + \lambda_a \cdot b - \lambda_a\lambda_b \cdot e = (a - \lambda_a \cdot e)b + \lambda_a(b - \lambda_b \cdot e) = 0$$

This shows that $h(ab) = h(a)h(b)$. Hence, h is an algebra morphism. Clearly, h is a bijective. Moreover, $a - \lambda_a \cdot e = 0$ implies that $|\lambda_a| = \|a\|$. Hence, h is an isometric algebra isomorphism. □

Remark 6.5. *Most of the results presented above hold for a non-unital Banach algebra by passing to its unitization. For example, if* A *is a non-unital algebra,* A' *is its unitization,* $a \in A$ *and* $p \in \mathbb{C}[z]$, *then*

$$\sigma_A(p(x)) := \sigma_{A'}(p(x)) = p(\sigma_{A'}(x)) := p(\sigma_A(x))$$

by the proof of the polynomial spectral mapping theorem.

7. Gelfand Transform

The Gelfand transform is a fundamental tool in the theory of Banach algebras, providing a powerful method for analyzing the structure of commutative algebras. It is crucial in the study of the spectrum of operators and in representation theory. The Gelfand transform also plays a central role in understanding the duality between a Banach algebra and its space of continuous functions. The Gelfand transform is also the first step in establishing that the category of C^*-algebras is equivalent to the category of locally compact Hausdorff spaces.

How should one study an arbitrary unital commutative Banach algebra? A key tenet of modern mathematics is that an abstract object should be studied by examining the algebra of continuous functions on it.

Definition 7.1. Let A be a Banach algebra. The **character space**, denoted as $\widehat{A}$, is the set of all non-zero linear maps,

$$\omega : A \to \mathbb{C},$$

that are also group homomorphisms. Each such ω is called a **character**.

Lemma 7.2. *Let A be a unital Banach algebra, and let $\widehat{A}$ be its character space. The following statements are true:*

1. *If $\omega \in \widehat{A}$, then $\omega(e) = 1$.*
2. *If $x \in \mathrm{GL}(A)$, then $\omega(x) \neq 0$.*
3. *If $\omega \in \widehat{A}$, then*
$$|\omega(a)| \leq \|a\|$$
for each $a \in A$. In particular, ω is continuous.
4. *If A is commutative, then there is a bijective correspondence between $\widehat{A}$ and the set of all two-sided maximal ideals in A (which are closed in A).*
5. *For every $\lambda \in \sigma_A(a)$, there is a character $\omega \in \widehat{A}$ such that $\omega_\lambda(a) = \lambda$.*

Proof. The proof is given below:

1. Let $a \in A$ such that $\omega(a) \neq 0$. Then,
$$\omega(a) = \omega(ae) = \omega(a)\omega(e)$$
This implies that
$$\omega(a)(1 - \omega(e)) = 0$$
Since $\omega(a) \neq 0$, we must have that $\omega(e) = 1$.
2. We have
$$\omega(x^{-1})\omega(x) = \omega(x^{-1}x) = \omega(e) = 1$$
Hence $\omega(x) \neq 0$.
3. For each $a \in A$, we have that $\sigma_A(a) \subseteq B(0, \|a\|)$. Therefore, if $|z| > \|a\|$, then $a - ze$ is invertible. Hence,
$$\omega(a) - z = \omega(a) - z\omega(e) = \omega(a - ze) \neq 0$$
Therefore, if $|z| > \|a\|$, then we cannot have that $\omega(a) = z$. Hence,
$$|\omega(a)| \leq \|a\|$$
In particular, this implies that ω is continuous.

(4) Let $\omega \in \widehat{A}$. Clearly, $\ker \omega$ is a closed subsapce since ω is continuous. Since ω is multiplicative, $\ker \omega$ is a two-sided ideal. Since the kernel of every linear map into $\mathbb{C}$ has codimension one, $\ker \omega$ is a maximal ideal. Conversely, let I be a two-sided maximal ideal of A. Proposition 4.2 implies that A/I is a Banach algebra. It is a standard algebraic fact that every element in A/I is invertible. Hence, Corollary 6.4 implies that $A/I \cong \mathbb{C}$. Hence, there is a homomorphism $\psi : A/I \to \mathbb{C}$. We can define a map $\omega : A \to \mathbb{C}$ by $\omega = \psi \circ \tau$, where τ is the canonical projection map. This map is clearly linear, since τ and ψ are. Also,

$$\begin{aligned} \omega(a)\omega(b) &= \psi(\tau(a))\psi(\tau(b)) \\ &= \psi(\tau(a)\tau(b)) \\ &= \psi(\tau(ab)) \\ &= \omega(ab), \end{aligned}$$

Therefore, ω is multiplicative; it is nonzero because $\omega(b) \neq 0$ for each $b \notin I$. Hence $\omega \in \widehat{A}$. Finally, $I \subseteq \ker(\omega)$ since $I = \ker(\tau)$; but if $b \notin I$ we know that $\omega(b) \neq 0$. Hence, $I = \ker \omega$. This also shows that I is closed.

(5) Since $a - \lambda e$ is not invertible, it generates a proper ideal $I = \langle a - \lambda e \rangle$ in A. I is contained in a maximal ideal, M, which is the kernel of some character $\omega_\lambda \in \widehat{A}$. Moreover,

$$\omega_\lambda(a) = \lambda \iff \lambda - ae \in \ker \omega_\lambda \iff \lambda - ae \in M$$

The last condition is true. Hence, $\omega_\lambda(a) = \lambda$.

This completes the proof. □

Remark 7.3. *Let A be a non-unital Banach algebra. In this case, the proof of Lemma 7.2 can be modified. For instance, we can still show that for each $\omega \in \widehat{A}$, we have that $\|\omega\| \leq 1$. For every $a \in A$ and $n \in \mathbb{N}$, we have*

$$|\omega(a)| = |\omega(a^n)|^{1/n} \leq \|\omega\|^{1/n}\|a^n\|^{1/n}$$

Thus

$$|\omega(a)| \leq \limsup_{n\to\infty} \|\omega\|^{1/n}\|a^n\|^{1/n} = r(a) \leq \|a\|[3].$$

Therefore, $\|\omega\| \leq 1$. Recall that a left (resp. right) ideal $I \subseteq A$ is called regular if A/I contains a unit. We can show that there is bijection between $\widehat{A}$ and the set of all regular two-sided maximal ideals. The proof in Proposition 5.11(3) now goes through since we can apply the Gelfand-Mazur theorem under the assumption that I is a maximal regular ideal.

Remark 7.4. *If A is a commutative Banach algebra, Lemma 7.2 implies that $\widehat{A} \neq \emptyset$ since Zorn's lemma guarantees the existence of at least one non-zero maximal ideal if A is unital, and the existence of a non-zero maximal regular ideal if A is non-unital. In order to produce a non-trivial maximal ideal, we have to assume that A is a commutative Banach algebra Indeed, the statement above does not hold if $A = M(2, \mathbb{C})$. This follows because $M(2, \mathbb{C})$ is a simple ring.*

[3]Note that the spectral radius is defined by passing to the unitzation of A.

If A is a (unital) commutative Banach algebra, Lemma 7.2 also implies that

$$\widehat{A} \subseteq A^*,$$

where A^* is the dual space of A. Recall that A^* is usually considered endowed with the weak-$*$ topology. Hence, the topology on $\widehat{A}$ is the relative weak-$*$ topology. In fact, we can say a bit more about the topological properties of $\widehat{A}$.

Proposition 7.5. *Let A be a unital commutative Banach algebra. Then $\widehat{A}$ is a compact Hausdorff subspace of A^* in the relative weak-$*$ topology on $\widehat{A}$.*

Proof. Clearly, $\widehat{A}$ is a weak-$*$ Hausdorff space since A^* is a weak-$*$ Hausdorff space. We now show that $\widehat{A}$ is weak-$*$ closed. Let $(\omega_n)_{n\in\mathbb{N}} \subseteq \widehat{A}$ such that $\omega_n \to \omega$ in the weak-$*$ topology for some $\omega \in A$. That is, $\omega_n(a) \to \omega(a)$ for all $a \in A$. We have,

$$\begin{aligned}
|\omega(ab) - \omega(a)\omega(b)| &= |\omega(ab) - \omega_n(ab) + \omega_n(a)\omega_n(b) - \omega(a)\omega(b)| \\
&\leq |\omega(ab) - \omega_n(ab)| + |\omega_n(a)\omega_n(b) - \omega(a)\omega(b)| \\
&\leq |\omega(ab) - \omega_n(ab)| + |(\omega_n(a) - \omega(a))\omega_n(b) + \omega(a)(\omega_n(b) - \omega(b))| \\
&\leq |\omega(ab) - \omega_n(ab)| + |\omega_n(a) - \omega(a)|\|b\| + \|a\||\omega_n(b) - \omega(b)|.
\end{aligned}$$

Since $\omega_n \to \omega$ in the weak-$*$ topology, we obtain that

$$|\omega(ab) - \omega(a)\omega(b)| = 0$$

Hence, $\omega \in \widehat{A}$. Hence, $\widehat{A}$ is weak-$*$ closed. By Lemma 7.2(2), we have $\widehat{A}$ is contained in the unit ball in A^*. By the Banach-Alaoglu theorem, the unit ball in A^* is weak-$*$ compact. Since, $\widehat{A}$ is weak-$*$ closed set of a weak-$*$ compact set, $\widehat{A}$ is a weak-$*$ compact set since A^* is Hausdorff. □

The motivation behind the Gelfand transform is that a Banach algebra, A, should be studied by invoking the principle of duality: elements in a Banach algebra can be studied can by studying the collection of evaluation maps. More precisely, we consider the map

$$\begin{aligned}
\widetilde{\Gamma} : A &\to A^{**} \\
a &\mapsto \widetilde{\Gamma}(a)(\phi) = \phi(a)
\end{aligned}$$

When $\omega \in \widehat{A}$, this defines $\widetilde{\Gamma}(a)$ as a function on $\widehat{A}$ for each $a \in A$. The definition of the weak-$*$ topology implies that $\widetilde{\Gamma}(a) \in C(\widehat{A})$.

Definition 7.6. Let A be a unital commutative Banach algebra. The map Γ defined as,

$$\begin{aligned}
\Gamma : A &\to C(\widehat{A}) \\
a &\mapsto \Gamma(a)(\omega) = \omega(a)
\end{aligned}$$

is the **Gelfand transform**.

Proposition 7.7. *Let A be a unital commutative Banach algebra, and let $\Gamma : A \to C(\widehat{A})$ denote the Gelfand transform.*

(1) *The Gelfand transform is an algebra homomorphism.*
(2) *The spectrum of $a \in A$ is:*

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\}$$

(3) *The Gelfand transform is a contraction, that is,*

$$\|\Gamma(a)\|_\infty \leq \|a\|.$$

for each $a \in A$.

Proof. The proof is given below:

(1) This is clear.
(2) For $a \in A$ and $z \in \mathbb{C}$, consider the element $a - ze$. If $a - ze$ is invertible, then $\omega(a - ze) \neq 0$ for each $\omega \in \widehat{A}$. If $a - ze$ is not invertible, then $a - ze$ is contained in a proper maximal ideal of A. Invoking Lemma 7.2(3), we have that there is a $\omega \in \widehat{A}$ such that $\omega(a - ze) = 0$. Hence $a - ze$ is invertible if and only if $\omega(a - ze) \neq 0$ for all $\omega \in \widehat{A}$ if and only if $\omega(a) \neq \omega(z)$ for all $\omega \in \widehat{A}$. Hence,

$$\rho_A(a) = \{z \in \mathbb{C} \mid z \neq \omega(a) \text{ for all } \omega \in \widehat{A}\}$$

Taking the complement, we have,

$$\begin{aligned}\sigma_A(a) &= \{z \in \mathbb{C} \mid z = \omega(a) \text{ for some } \omega \in \widehat{A}\} \\ &= \{\omega(a) \mid \omega \in \widehat{A}\}\end{aligned}$$

(3) We have,

$$\|\Gamma(a)\|_\infty = \sup_{\omega \in \widehat{A}} |\omega(a)| = r(a) \leq \|a\|$$

This completes the proof. □

Remark 7.8. *Note that $x \in \mathrm{GL}(A)$ if and only if $\Gamma(a)$ never vanishes. Observe that*

$$\begin{aligned} a \textit{ is not invertible} &\iff I = \langle a \rangle \textit{ is proper} \\ &\iff a \textit{ is contained in a maximal ideal} \\ &\iff \omega(a) = 0 \textit{ for some } \omega \in \widehat{A} \\ &\iff \Gamma(a) \textit{ has a zero.}\end{aligned}$$

There are examples non-unital Banach algebras. It turns out that we can extend the results discussed to the case of non-unital Banach algebras.

Proposition 7.9. *Let A be a non-unital commutative Banach algebra and let A' denote its unitization. Let Γ denote the Gelfand transform as in Proposition 7.7.*

(1) *We have,*

$$\widehat{A'} = \widehat{A} \cup \{\phi_0\},$$

where we have define $\phi_0((a, \lambda)) = \lambda$

(2) *$\widehat{A}$ is a locally compact, Hausdorff space.*
(3) *The Gelfand transform is a contractive, algebra homomorphism into $C_0(\widehat{A})$.*
(4) *The spectrum of $a \in A$ is*

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\} \cup \{0\}$$

for each $a \in A$.

Proof. (Sketch) The proof is given below:

(1) It is clear that there can be no other elements of $\widehat{A'}$. If $\phi \in \widehat{A'}$, then the restriction of ϕ to A has a unique multiplicative extension to A' (see Proposition 11.6), unless it identically vanishes on A. In the latter case, ϕ_0 is clearly the only possibility.

(2) Since $\widehat{A'}$ is a compact Hausdorff space, $\widehat{A} = \widehat{A'} \setminus \{\phi_0\}$ is a locally compact, Hausdorff space by the characterization of locally compact, Hausdorff spaces.
(3) Note that $\Gamma(a)(\phi_0) = \phi_0(a) = 0$ This means that $\Gamma(A) \subseteq C_0(\hat{A})$.
(4) We have,

$$\sigma_A(a) := \sigma_{A'}(a) = \{\omega(a) \mid \omega \in \widehat{A'}\} = \{\omega(a) \mid \omega \in \widehat{A}\} \cup \{0\}$$

This completes the proof. □

Remark 7.10. *If A is a (unital) commutative Banach algebra, we have proved that the Gelfand transform is a contractive (hence injective) algebra homomorphism. The Gelfand transform becomes an isometric $*$-isomorphism if A is a commutative C^*-algebra.*

We end with some examples. We first compute $\widehat{C(X)}$ when X is a compact Hausdorff space.

Example 7.11. Let X be a compact Hausdorff space. We find $\widehat{C(X)}$. By Lemma 7.2(3), it suffices to compute the set of maximal ideals of $C(X)$. For each $x \in X$, define I_x by

$$I_x = \{f \in C(X) : f(x) = 0\}$$

Let I be a *proper* ideal in $C(X)$. We claim that $I \subseteq I_x$ for some $x \in X$. Assume this is not the case. Then for each $x \in X$, we can find a $f_x \in I$ such that $f_x \notin I_x$. That is, $f_x(x) \neq 0$. Since f_x is continuous there is a open neighbourhood $x \in U_x$ that $f_x|_{U_x} \neq 0$. Since X is compact, the open cover $(U_x)_{x\in X}$ admits a finite sub-cover. Hence, we can find $x_1, \dots, x_n \in X$ such that $X = \cup_{i=1}^n U_{x_i}$. Consider the function

$$f(x) = \sum_{i=1}^n |f_{x_i}(x)|^2 = \sum_{i=1}^n \overline{f_{x_i}(x)} f_{x_i}(x)$$

Clearly, $f \in I$. But by construction $f > 0$ on X. Hence, f is invertible, so that I contains an invertible element, contradicting that I is a proper ideal. Hence, we have that for every proper ideal, I, there exists a $x_I \in I$ such that $I \subseteq I_{x_I}$. Moreover, let $x \neq y$. Since X is compact and Hausdorff, X is a normal space. By Urysohn's lemma, there exists $f, g \in C(X)$ such that $f|_{\{x\}} = 0, f|_{\{y\}} = 1$, and $g|_{\{y\}} = 1, g|_{\{y\}} = 0$. This shows that $I_x \subsetneq I_y$ and $I_y \subsetneq I_x$. Hence, we can conclude that the set of maximal ideas, $\mathscr{M}$. is of the form

$$\mathscr{M} = \{I_x \mid x \in X\}$$

In particular, we can conclude that $\widehat{C(X)} = X$ as sets. In fact the map

$$\varphi : X \to \widehat{C(X)}$$
$$x \mapsto \mathrm{Ev}_x$$

is a homemorphism. It is clear that each $\mathrm{Ev}_x \in \widehat{C(X)}$, and $\mathrm{Ev}_x \neq \mathrm{Ev}_x$ for $x \neq y$ as above. If $x_\alpha \to x$, then $f(x_\alpha) \to f(x)$ for each $f \in C(X)$, which implies that $\mathrm{Ev}_{x_\alpha} \to \mathrm{Ev}_x$ in the weak-$*$topology. This shows that φ is a continuous injection. It is also surjective by the discussion above. Since these spaces are compact Hausdorff, the claim follows.

In fact, one can generalize Example 7.11.

Example 7.12. Let X be a locally compact Hausdorff space. We claim that

$$\begin{aligned} \varphi : X &\to \widehat{C_0(X)} \\ x &\mapsto \mathrm{Ev}_x \end{aligned}$$

is a homeomorphism. Continuity follows as in Example 7.11. Injectivity follows as in Example 7.11 by applying Uryhson's lemma to X_∞. We show φ is onto. Let $\omega \in \widehat{C_0(X)}$. By the Riesz representation theorem , there exists a positive Radon measure μ on X such that

$$\omega(f) = \int_X f(x)\, d\mu(x) \quad \text{for all } f \in C_0(X).$$

Thus, we have

$$0 = \omega\left(\overline{(f - \omega(f))}(f - \omega(f))\right) = \int_X |f(x) - \omega(f)|^2 d\mu(x).$$

This means that, for every $f \in C_0(X)$, f equals the constant function $\omega(f)$ μ-almost everywhere. Hence, there is a point $x_0 \in X$ such that

$$\omega(f) = f(x_0) \quad \text{for all } f \in C_0(X).$$

In other words, $\omega = \mathrm{Ev}_{x_0}$. One can φ to a continuous and bijective map from X_∞ onto $\widehat{A'}$. This is a homemorphism as in Example 7.11. Hence, so is φ.

Classical Mathematics	Quantum Mathematics
Groups	Quantum groups
Cohomology	Quantum Cohomology
Topology	C^*-algebras
Differential Geometry	Non Commutative Geometry
Probability Theory	Free Probability
Information Theory	Quantum Information Theory

Correspondence of some topics in *classical mathematics* and *quantum mathematics.*

Part 2. C^*-**Algebras I: Basic Theory**

8. Why C^*-Algebras?

Linear algebra studies linear operators which are linear maps $T : \mathbb{C}^n \to \mathbb{C}^n$. If $T, S : \mathbb{C}^n \to \mathbb{C}^n$ are two such linear operators, then $T \circ S \neq S \circ T$ in general. This is perhaps the first instance where one encounters the phenomenon of 'non-commutativity' in mathematics. More generally, functional analysis studies infinite-dimensional linear spaces with additional analytic structures. A key example is that of a Hilbert spaces. If $\mathscr{H}$ is an infinite-dimensional Hilbert space, we consider the Hilbert space of bounded/continuous linear operator on $\mathscr{H}$:

$$\mathscr{B}(\mathscr{H}) := \{T : H \to H \mid T \text{ is linear and bounded}\},$$

Once again, elements of $\mathscr{B}(\mathscr{H})$ are non-commutative, in general. The phenomenon of non-commutativity is prevalent in different topics of mathematics and physics. Examples include quantum physics, linear algebra, representation theory of groups, etc. The theory of operator algebras (including C^*-algebras) attempts to capture the essence of non-commutativity. The pioneers of operator algebras, Francis Murray and John von Neumann, wrote in their very first article on operator algebras[4] in 1936 that

"various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject."

Hence, the study of operator algebras can be considered an essential part of *non-commutative mathematics*, which is also called *quantum mathematics*[5] more colloquially and popularly.

Why C^*-algebras, though? C^*-algebras can be thought of as a non-commutative or quantum version of topology. This is the content of a result of Gelfand and Naimark (to be proved later):

Theorem 8.1. *(Gelfand & Naimark) Let A be a C^*-algebra. Then A is commutative if and only if A is isometrically $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space, X.*

Corollary 8.2. *Let A be a unital C^*-algebra. Then A is commutative if and only if A is isometrically $*$-isomorphic to $C(X)$ for some compact Hausdorff topological space, X.*

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[4] It was an article on von Neumann algebras

[5] Thus, phrases such as *quantum groups*, *quantum cohomology*, etc., are used to describe objects studied in *quantum mathematics.*

Hence, any locally compact Hausdorff topological space gives rise to a commutative C^*-algebra. On the other hand, any commutative C^*-algebra is exactly of this form. In this sense, commutative C^*-algebras correspond to "commutative-topology," and we may view the theory of non-commutative C^*-algebras as a kind of "non-commutative topology." This duality is also the basis for other topics in non-commutative/quantum mathematics. Refer to Table 1.

9. $*$-Algebras

We first define $*$-algebras. The $*$-operation on a algebra is analogous to taking adjoints in complex matrix algebras.

Definition 9.1. Let V be an $\mathbb{C}$-algebra. Then V is a **$*$-algebra** if there is a map $* : A \to A$ is an antiautomorphism and an involution. More precisely, $*$ is required to satisfy the following properties:

(1) $(x+y)^* = x^* + y^*$,
(2) $(xy)^* = y^* x^*$,
(3) $(x^*)^* = x$,
(4) $(kx)^* = \overline{k} x^*$

for all $x, y \in V$ and $k \in \mathbb{C}$.

A Banach $*$-algebra is a Banach algebra with a $*$ map.

Definition 9.2. A **Banach $*$-algebra** is a Banach algebra that is a $*$-algebra. A morphism $\phi : A \to B$ between two Banach $*$-algebra A and B is a morphism of the underlying Banach algebras that that is $*$-preserving. That is, $\phi(a^*) = \phi(a)^*$ for each $a \in A$.

We end with some properties of Banach $*$-algebra.

Proposition 9.3. *Let A be a unital Banach $*$-algebra.*

(1) $e = e^*$.
(2) *If $x \in \mathrm{GL}(A)$, then $x^* \in \mathrm{GL}(A)$, and $(x^*)^{-1} = (x^{-1})^*$.*
(3) $\sigma_A(a^*) = \overline{\sigma_A(a)}$ *for each $a \in A$.*

Proof. The proof is given below:

(1) By definition, we have $e^* e = e^*$. Applying $*$, this implies $e^* = e^* e = e$.
(2) This is clear.
(3) By (1)
$$a^* - \lambda e = a^* - (\overline{\lambda} e)^* = (a - \overline{\lambda} e)^*$$
The claim follows by (2) now.

This completes the proof. □

10. C^*-Algebras

We have seen that the canonical example of a non-commutative Banach algebra if $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is some Hilbert space. It turns out that $\mathscr{B}(\mathscr{H})$ has even more structure. Recall that the if $T \in \mathscr{B}(\mathscr{H})$, the adjoint of T, denoted as T^*, is in $\mathscr{B}(\mathscr{H})$ defined by the property
$$\langle \Psi, T^* \Phi \rangle_{\mathscr{H}} := \langle T\Psi, \Phi \rangle_{\mathscr{H}}$$
for all $\Psi, \Phi \in \mathscr{H}$. For $T, S \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \mathbb{C}$, it is well-known that the adjoint operator satisfies the following properties:

(1) $(T+S)^* = T^* + S^*$;
(2) $T^{**} = T$;
(3) $(TS)^* = S^*T^*$;
(4) $(\lambda T)^* = \overline{\lambda} T^*$.

Remark 10.1. *Since the adjoint operation, denoted as $*$, is idempotent ($T^{**} = T$), we say that $*$ is an involution.*

This makes $\mathscr{B}(\mathscr{H})$ into a Banach $*$-algebra. How does the adjoint operation interact with the norm? Pick $\Psi \in H$, and use the Cauchy-Schwarz inequality to estimate

$$\|T\Psi\|^2 = \langle T\Psi, T\Psi \rangle = \langle \Psi, T^*T\Psi \rangle \leq \|\Psi\| \|T^*T\Psi\| \leq \|T^*T\| \|\Psi\|^2.$$

Using the definition of the operator norm and the sub-multiplicative nature of the operator norm, we infer that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \|T\|. \tag{1}$$

This implies to $\|T\| \leq \|T^*\|$. Replacing T by T^* and the fact that $T^{**} = T$ implies $\|T^*\| \leq \|T\|$. Hence,

$$\|T^*\| = \|T\|,$$

for each $T \in \mathscr{B}(\mathscr{H})$. Substituting this in Equation (1), we derive the crucial property

$$\|T^*T\| = \|T\|^2 \tag{2}$$

for each $T \in \mathscr{B}(\mathscr{H})$. This is called that C^* identity. This discussion motivates the definition of a C^*-algebra.

Definition 10.2. A **C^*-algebra**, A, is a Banach $*$-algebra that is a $*$-algebra that satisfies the C^* identity:

$$\|a^*a\| = \|a\|^2$$

for each $a \in A$. A morphism $\phi : A \to B$ between two C^* algebras A and B is a morphism of the underlying Banach $*$-algebras.

Example 10.3. The following is a list of some basics examples of C^* algebras:

(1) As discussed above, $\mathscr{B}(\mathscr{H})$ is a C^* algebra for any Hilbert space, $\mathscr{H}$.
(2) If X is a locally compact Hausdorff topological space, then $C_0(X)$ is a C^* algebra with involution given by complex conjugation. This is an example of a non-unital C^* algebra.
(3) If X is a locally compact Hausdorff topological space, $C_b(X)$ is also a C^* algebra with involution given by complex conjugation.

Remark 10.4. *More interesting examples and constructions will be discussed later on.*

Analogously with the algebraic characterization of operators in $\mathscr{B}(\mathscr{H})$, we have the following special class of elements a in a C^*-algebra, A

- a is normal if $a^*a = aa^*$,
- a is self-adjoint if $a = a^*$,
- a is a projection if $a = a^* = a^2$,
- a is a unitary if $a^*a = aa^* = 1$,
- a is an isometry if $a^*a = 1$,

Proposition 10.5. *Let A be a non-zero C^*-algebra. The following is a list of some elementary properties of a C^*-algebra.*

(1) *$\|a\| = \|a^*\|$ for each $a \in A$,*
(2) *A Banach $*$-algebra is a C^*-algebra if and only if $\|a\|^2 \leq \|a^*a\|$ for each $a \in A$*
(3) *If A is unital and e is the identity element, then $\|e\| = 1$.*
(4) *Any element a in a C^*-algebra is the sum of two self-adjoint operators.*
(5) *If A is unital and $a \in A$ is unitary, then $\sigma_A(a) \subseteq \mathbb{S}^1$.*
(6) *If A is unital, and $a \in A$ is a self-adjoint element, then $r(a) = \|a\|$.*
(7) *If A is unital, and $a \in A$ is a normal element, then $r(a) = \|a\|$.*
(8) *If A is unital, then the norm on A is unique.*
(9) *A linear map between C^*-algebras is $*$-preserving if and only if it maps self-adjoint elements to self-adjoint elements.*
(10) *Let A, B be two unital C^* algebras. A $*$-homomorphism $\pi : A \to B$ is contractive (i.e. $\|\pi(a)\| \leq \|a\|$ for each $a \in A$) and hence continuous. Moreover, a $*$-isomorphism between C^* algebras is isometric.*
(11) *If $\phi \in \widehat{A}$ and $a \in A$ is self-adjoint, then $\phi(a) \in \mathbb{R}$.*
(12) *If $\phi \in \widehat{A}$, then ϕ is $*$-preserving with $\|\phi\| = 1$.*
(13) *If $\phi \in \widehat{A}$, then $\phi(a^*a) \geq 0$.*
(14) *If A is unital, and $\phi \in \widehat{A}$, and a is a unitary, then $|\phi(a)| = 1$.*

Proof. The proof is given below:

(1) We use the C^*-identity. We have,
$$\|a\|^2 = \|aa^*\| \leq \|a\|\|a^*\|$$
Hence, we have $\|a\| \leq \|a^*\|$. Similarly, we have $\|a^*\| \leq \|a^{**}\| = \|a\|$. The result now follows.

(2) If A is a Banach $*$-algebra satisfying the given assumption, we have,
$$\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\| \leq \|a\|\|a\|$$
The last inequality follows since (1) holds in a Banach $*$-algebra. Hence, we have an equality above, which implies that the C^* identity holds. The converse is clear.

(3) The C^* identity implies that
$$\|e\| = \|ee\| = \|e^*e\| = \|e\|^2$$
Hence, $\|e\| = 1$[6].

(4) Simply note that, $a = \mathrm{Re}(a) + i\,\mathrm{Im}(a)$, where
$$\mathrm{Re}(a) = \frac{1}{2}(a + a^*) \qquad \mathrm{Im}(a) = \frac{1}{2i}(a - a^*)$$
It is clear that $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are self-adjoint elements.

(5) Recall that for any invertible element, a, in a Banach algebra,we have
$$\sigma_A(a^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma_A(a)\}$$
Since a and a^* are unitary, we have,
$$\|a\|^2 = \|a^*a\| = \|e\| = 1 = \|e\| = \|aa^*\| = \|a^*\|^2,$$
Hence, $\|a\| = \|a^*\| = 1$. Therefore, if $\lambda \in \sigma_A(a)$, we have $|\lambda| \leq 1$. Similarly, for any $\lambda \in \sigma_A(a)$, we have that $\lambda^{-1} \in \sigma_A(a^{-1}) = \sigma_A(a^*)$. Hence, $|\lambda^{-1}| \leq 1$. Hence, $|\lambda| = 1$.

[6]Note that $\|e\| \neq 0$ since $e \neq 0$ since A is a non-zero C^* algebra.

(6) The C^*-identity implies that

$$\|a\|^2 = \|a^*a\| = \|a^2\|$$

Repeated use of the C^* identity implies that,

$$\|a\|^{2^n} = \|a^{2^n}\|$$

Hence,

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} \|a\|^{\frac{2^n}{2^n}} = \|a\|.$$

(7) The C^* identity implies that,

$$\|a^2\|^2 = \|(a^2)(a^2)^*\| = \|(a^*a)^*(a^*a)\| = \|a^*a\|^2 = (\|a\|^2)^2$$

holds. The remaining argument is same as in (8).

(8) If $a \in A$ is any element, then,

$$\|a\|^2 = \|a^*a\| = r(a^*a)$$

Hence, we see that the spectral radius *intrinsically* determines the norm of any element in a C^* algebra. Hence, the norm is uniquely defined.

(9) A $*$-preserving map clearly sends self-adjoint elements to self-adjoint elements. Conversely, let $\phi : A \to B$ be a linear map that maps self-adjoint elements to self-adjoint elements. Using (5), we have,

$$\phi(a) = \phi(\mathrm{Re}(a)) + i\phi(\mathrm{Im}(a))$$

$$\phi(a^*) = \phi(\mathrm{Re}(a)) - i\phi(\mathrm{Im}(a)).$$

Since $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are self-adjoint, $\phi(\mathrm{Re}(a))$ and $\phi(\mathrm{Im}(a))$ are self-adjoint by assumption. So

$$\begin{aligned}\phi(a)^* &= \phi(\mathrm{Re}(a) + i\,\mathrm{Im}(a))^* \\ &= (\phi(\mathrm{Re}(a)) + i\phi(\mathrm{Im}(a)))^* \\ &= \phi(\mathrm{Re}(a)) - i\phi(\mathrm{Im}(a)) \\ &= \phi(a^*).\end{aligned}$$

(10) Let $a \in A$. Then a^*a is a normal element in A, which means $\|a^*a\| = r(a^*a)$ by (9). By Proposition 5.7(3) , we have, $r(\pi(a^*a)) \leq r(a^*a)$. Hence,

$$\begin{aligned}\|a\|^2 &= \|a^*a\| \\ &= r(a^*a) \\ &\geq r(\pi(a^*a)) \\ &= r(\pi(a^*)\pi(a)) \\ &= \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.\end{aligned}$$

If π is a $*$-isomorphism, then a symmetric argument shows the inequality above is an equality.

(11) Let $a = a^*$, and let $\phi(a) = \alpha + i\beta \in \mathbb{C}$. Note that $\phi(a) + i\lambda = \phi(a + i\lambda)$ and $|\phi(x) + i\lambda| \leq \|a + i\lambda\|$ for all $\lambda \in \mathbb{R}$. Hence:

$$\begin{aligned}\alpha^2 + (\lambda + \beta)^2 &= |\phi(a) + i\lambda|^2 \\ &\leq \|a + i\lambda\|^2 \\ &= \|(a + i\lambda)^*(a + i\lambda)\| \\ &= \|a^2 + \lambda^2\| \\ &\leq \|a\|^2 + \lambda^2\end{aligned}$$

Thus, $\alpha^2 + 2\lambda\beta + \beta^2 \leq \|a\|^2$, for all $\lambda \in \mathbb{R}$, which implies $\beta = 0$.

(12) We already know that $\|\phi\| \leq 1$. But $\phi(e) = 1$ implies that $\|\phi\| = 1$. The $*$-preserving condition follows from (10) and (12).

(13) We have,

$$\phi(a^*a) = \phi(a^*)\phi(a) = \overline{\phi(a)}\phi(a) \geq 0$$

The last equality follows by (13).

(14) We have,

$$|\phi(a)|^2 = \overline{\phi(a)}\phi(a) = \phi(a^*)\phi(a) = \phi(a^*a) = \phi(e) = 1$$

This completes the proof. □

Let's end with a non-example of a C^*-algebra.

Example 10.6. Let $\mathscr{A}(\mathbb{D})$ be the disk algebra. It is clear that $\mathscr{A}(\mathbb{D})$ is a Banach $*$-algebra with the $*$-operation given by

$$f^*(z) := \overline{f(\bar{z})}$$

We show that $\mathscr{A}(\mathbb{D})$ is not a C^*-algebra. Let $f(z) = e^{iz} \in \mathscr{A}(\mathbb{D})$. We have $f^*(z) = e^{-i\bar{z}}$. For $z \in \mathbb{D}$

$$f^*f(z) := f^*(z)f(z) = e^{-i\bar{z}}e^{iz} = e^{i(z-\bar{z})} = 1$$

Therefore $\|f^*f\| = 1$. On the other hand, we have

$$\begin{aligned}\|f\|^2 &= \sup\{|e^{iz}|^2 : z \in \mathbb{D}\} \\ &= \sup\{e^{-i\bar{z}+iz} : z \in \mathbb{D}\} \\ &= \sup\{e^{-2\operatorname{Im} z} : z \in \mathbb{D}\} \\ &= \sup\{e^{-2b} : b \in [-1, 1]\} \\ &= e^2.\end{aligned}$$

Hence, the C^*-identity is not satisfied.

11. Unitization

Not all C^*-algebras have units. The primary example is $C_0(\mathbb{R})$. In this case, we can consider the unitization of a non-unital C^* algebra.

Definition 11.1. Let A be a non-unital C^*-algebra. The (smallest) unital C^*-algebra containing A is called its **unitization**, A'. defined A' as follows:

$$A' := A \oplus \mathbb{C}$$

with algebraic operations given by

$$(a, \alpha) \cdot (b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$$
$$(a, \alpha)^* = (a^*, \overline{\alpha})$$
$$\|(a, \alpha)\| = \sup_{b \in A, \|b\| \leq 1} \|ab + \alpha b\|$$

Remark 11.2. *Consider the map,*

$$\Phi : A \to \mathscr{B}(A) \qquad a \mapsto L_a,$$

where L_a is the left-multiplication operator on A. It is easy to check that Φ is a $*$-homomorphism. We have that $\|L_a\| = \|a\|$. Indeed, if $x \in A$ such that $\|x\| \leq 1$, then

$$\|ax\| \leq \|a\|\|x\|$$

Hence,

$$\|L_a\| = \sup_{\|x\| \leq 1} \|ax\| \leq \sup_{\|x\| \leq 1} \|a\|\|x\| = \|a\|$$

If $x = a^*/\|a\|$, then $\|x\| = 1$, and

$$L_a(x) = \|aa^*/a\| = \|a\|$$

This shows that Φ is an isometric $*$-homomorophism. Hence, A can be identified with a $*$-subalgebra of $\mathscr{B}(A)$. If we identify $a \in A$ with the left multiplication operator $L_a \in \mathscr{B}(A)$, and we identify (a, α) with the operator $L_a + \alpha \, \mathrm{Id}_A$, then the norm on A' is the norm induced from $\mathscr{B}(A)$ on the $*$-subalgebra $\langle L_a, Id \mid a \in A \rangle$.

Remark 11.3. *We have already seen that the norm on a C^* algebra is unique. Thus, the norm defined above is the 'right' choice.*

Let's verify that A' is indeed a unital C^*-algebra. The unit is $(0, 1)$, and clearly A' is a$*$-algebra. It is a norm by the remark made above. Moreover, note that the identification $a \mapsto L_a$ is isometric. Indeed, using the C^*-identity in A, we have for any nonzero $a \in A$,

$$\|a\| = \left\| a\Big(\frac{a^*}{\|a\|}\Big) \right\| \leq \sup_{\|b\| \leq 1} \|ab\| \leq \|a\| \sup_{\|b\| \leq 1} \|b\| = \|a\|.$$

So, $\|(a, 0)\|_{A'} = \|a\|_A$, and the embedding of A into A' is isometric. Since $\mathscr{B}(A)$ is complete

$$\{L_a + \alpha \, \mathrm{id}_A : a \in A, \alpha \in \mathbb{C}\}$$

is complete since it is a closed subspace of $\mathscr{B}(A)$. Hence, A' is a Banach algebra.

It remains to show that the given norm satisfies the C^*-identity. To that end, we compute for $a \in A$ and $\alpha \in \mathbb{C}$, oh

$$\begin{aligned}
\|(a, \alpha)\|^2 &= \sup_{\|b\| \leq 1} \|ab + \alpha b\|^2 \\
&= \sup_{\|b\| \leq 1} \|b^*(a^*a + \alpha a^* + \overline{\alpha} a + |\alpha|^2 \, \mathrm{Id}_A) b\| \\
&\leq \sup_{\|b\| \leq 1} \|a^*a + \alpha a^* + \overline{\alpha} a + |\alpha|^2 \, \mathrm{Id}_A \| \\
&= \|(a, \alpha)^*(a, \alpha)\| \leq \|(a, \alpha)^*\| \|(a, \alpha)\|.
\end{aligned}$$

So $\|(a,\alpha)\| \leq \|(a,\alpha)^*\|$, and a symmetric argument yields $\|(a,\alpha)^*\| = \|(a,\alpha)\|$. Then the above inequality gives

$$\|(a,\alpha)\|^2 \leq \|(a,\alpha)^*(a,\alpha)\| \leq \|(a,\alpha)\|^2.$$

This proves the C^* identity in A'.

Remark 11.4. *We denote the identity element in A' as e'.*

Remark 11.5. *Properties of unital C^* algebras discussed in Proposition 10.5 can be extended to the case of non-unital C^* algebras by passing to the unitization of non-unital C^* algebras.*

One thing that makes unitizations nice to work with is that a $*$-homomorphism always has a unique and natural extension to the unitization.

Proposition 11.6. *Let A, B be C^*-algebras with B unital and A non-unital and $\pi : A \to B$ a $*$-homomorphism. Then there is a unique extension of π to a unital $*$-homomorphism $\tilde{\pi} : A' \to B$ given by*

$$\tilde{\pi}(a + \lambda e') = \pi(a) + \lambda e_B$$

Proof. We just need to check that the given formula is a $*$-homomorphism. Linearity and $*$-preserving are immediate. For $a, b \in A$ and $\lambda, \eta \in \mathbb{C}$, we compute

$$\begin{aligned}\tilde{\pi}(a + \lambda e')\tilde{\pi}(b + \eta e') &= (\pi(a) + \lambda e_B)(\pi(b) + \eta e_B) \\ &= \pi(ab) + \lambda\pi(b) + \eta\pi(a) + \lambda\eta e_B \\ &= \tilde{\pi}(ab + \lambda b + \eta a + \lambda\eta e').\end{aligned}$$

The uniqueness is forced by the fact that we require $\tilde{\pi}$ to be linear and $e' \mapsto e_B$. □

Remark 11.7. *Note that the proof of Proposition 11.6 works also when we have $\pi : A \to B$ with B non-unital. Moreover, we did not actually use the fact that π was $*$-preserving in the proof. Indeed, it suffices to assume that π is linear and multiplicative map. Moreover, essentially the same proof works if A and B are assumed to be Banach algebras.*

12. Gelfand-Naimark Theorem

We prove the Gelfand-Naimark Theorem for commutative C^*-algebras.

Theorem 12.1. *(Gelfand & Naimark) Let A be a C^*-algebra. Then A is commutative if and only if A is isometrically $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space, X.*

We have already observed that if A is considered as a Banach $*$-algebra, the Gelfand transform,

$$\Gamma : A \to C_0(\widehat{A})$$

defines a contractive (and hence continuous) algebra homomorphism. The purpose of the remainder of the section is to show that Γ is $*$-preserving surjective isometry.

Proof. We first show that Γ is an isometry. Since A is commutative, every element in A is normal. Hence,

$$\|\Gamma(a)\|_\infty = \sup_{\omega \in \widehat{A}} |\omega(a)| = r(a) = \|a\|$$

The last equality follows from Proposition 10.5(8). Hence, Γ is isometric and injective. We now show that Γ is $*$-preserving. By Proposition 10.5(10), it suffices to show that Γ maps self-adjoint elements to self-adjoint elements. But if $a \in A$ is any self-adjoint element, we have that

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\} \subseteq \mathbb{R}$$

by Proposition 10.5(12). Hence, range$(\Gamma(a)) \subseteq \mathbb{R}$, which means $\Gamma(a) = \overline{\Gamma(a)}$ is self-adjoint. This shows that Γ is $*$-preserving. We now show that Γ is surjective. We have that $\Gamma(A)$ is a $*$-subalgebra of $C_0(\widehat{A})$. $\Gamma(A)$ separates points. Indeed, if $\omega_1 \neq \omega_2 \in \widehat{A}$, there is some $a \in A$ such that $\omega_1(a) \neq \omega_2(a)$. Hence, $\Gamma(a)(\omega_1) \neq \Gamma(a)(\omega_2)$. Moreover, $\Gamma(A)$ vanishes no-where. Indeed, if $\omega \in \widehat{A}$, since ω is non-zero, then there is a $a \in A$ such that $\Gamma(a)(\omega) \neq 0$. The Stone–Weierstrass theorem[7] now implies that $\overline{\Gamma(A)} = C_0(\widehat{A})$. But since A is a closed set of itse,f Γ is a linear isometry and $C_0(\widehat{A})$ is a normed space, general Banach space theory shows that $\Gamma(A)$ is in fact closed[8]. Hence, $\Gamma(A) = C_0(\widehat{A})$. □

13. Continuous Functional Calculus

Recall that Proposition 6.1 we characterized the spectrum of an element of a Banach algebra obtained by applying a polynomial function to an element of a Banach algebra. This is an example of polynomial functional calculus, which deals with the study of polynomials of functions of elements in a Banach algebra. Note that polynomial functional calculus can be extended to holomorphic functional calculus in a Banach algebra. In this section, we establish a more general functional calculus for C^*-algebras: continuous functional calculus, which is a functional calculus which allows the application of a continuous function to normal elements of a C^*-algebra.

13.1. **Motivation.** We already know how to apply polynomial functional calculus for polynomials defined on the spectrum of any element a of a Banach algebra. If we wish to extend polynomial functional calculus to continuous functions defined on spectrum, it seems obvious to approximate a continuous function by polynomials according to the Stone-Weierstrass theorem, and insert the element into these polynomials and to show that this sequence of elements converges in A. In particular, we shall approximate continuous functions on the spectrum of an element by Laurent polynomials, i.e., by polynomials of the form

$$p(z, \overline{z}) = \sum_{k,l=0}^{N} c_{k,l} z^k \overline{z}^l \qquad c_{k,l} \in \mathbb{C}$$

Here, $\overline{z}$ denotes the complex conjugation, which is an involution on the complex numbers. To be able to insert a in place of z in this kind of polynomial, Banach $*$-algebras are considered, i.e., Banach algebras that also have an involution $*$, and a^* is inserted in place of $\overline{z}$. In order to obtain a homomorphism

$$\mathbb{C}[z, \overline{z}] \to A,$$

a restriction to normal elements, i.e., elements with $a^*a = aa^*$, is necessary, as the polynomial ring $\mathbb{C}[z, \overline{z}]$ is commutative. If

$$(p_n(z, \overline{z}))_n$$

[7]The complex version for locally compact Hausdorff spaces.

[8]The claim is that the image of a closed set under a linear isometry from a Banach space to a normed vector space is closed.

is a sequence of polynomials that converges uniformly on the spectrum of a to a continuous function f, the convergence of the sequence

$$(p_n(a, a^*))_{n\in\mathbb{N}}$$

in A to an element $f(a)$ must be ensured. A detailed analysis of this convergence problem shows that it is necessary to resort to C^*-algebras. These considerations lead to the so-called continuous functional calculus.

13.2. **Construction.** Let A be a C^* algebra. For any $a \in A$ that is a normal element, we write $C^*(a)$ for the C^*-algebra generated by a. This can be identified as the norm closure of the set of all polynomials in a, a^* with zero constant term, i.e.,

$$C^*(a) = \overline{\{p(a, a^*) \mid p \in \mathbb{C}[z_1, z_2], p(0, 0) = 0\}}.$$

When A is unital with unit e, $C^*(a, e)$ can be identified with the closure of the set of all polynomials on a, a^*

$$C^*(a, e) = \overline{\{p(a, a^*) \mid p \in \mathbb{C}[z_1, z_2]\}}.$$

Moreover, we denote as $C(\sigma_A(a))$ the C^*-algebra of continuous functions on $\sigma_A(a)$, the spectrum of a. Note that all of these C^* algebras are commutative.

Proposition 13.1. *Let A be a unital C^* algebra and let a be a normal element of A. Then there exists a unique $*$-isometric algebra isomomorphism*

$$\Phi_a \colon C(\sigma_A(a)) \to C^*(e, a)$$

with $\Phi_a(1_{\sigma_A(a)}) = e$ for $1_{\sigma_A(a)}(z) = 1$ and $\Phi_a(\mathrm{Id}_{\sigma_A(a)}) = a$ for the identity. The mapping Φ_a is called the continuous functional calculus of the normal element a. Usually it is suggestively set $f(a) := \Phi_a(f)$.

We first need to prove the following lemma.

Lemma 13.2. *Let A be a unital C^*-algebra, and let B be a C^*-subalgebra containing the identity of A. Then for all $b \in B$, we have*

$$\sigma_B(b) = \sigma_A(b).$$

Proof. Clearly, we have that have

$$\sigma_A(b) \subseteq \sigma_B(b),$$

and the reverse inclusion will follow if we can show that b is invertible in A, implies that it is already invertible in B. First assume that $b = b^*$. Let,

$$E = C^*(b, b^{-1}) \subseteq A,$$
$$D = C^*(e, b) \subseteq E \cap B.$$

We show that $E = D$. This readily implies that $b^{-1} \in B$. Since $(b^{-1})^* = b^{-1}$, E is the closure of the algebra generated by $\{b, b^{-1}\}$. In particular, E is a unital commutative C^*-algebra. Hence,

$$E \cong C(\widehat{E})$$

Let $D' \subseteq C(\widehat{E})$ denote the image of D under the Gelfand transform. D' is a closed[9] $*$-subalgebra of $C(\widehat{E})$. Furthermore, if $\phi, \psi \in \widehat{E}$ are such that $\phi \neq \psi$, we must have $\phi(a) \neq \psi(a)$ since otherwise ϕ, ψ agree on a, a^{-1} and hence everywhere. Since $\mathrm{Ev}_a \in D'$,

[9][Junaid:Why is it closed? Is D a $*$-closed subalgebra of E?]

D' separates the points of $\widehat{E}$. By Stone-Weierstrass, $\overline{D'} = C(\widehat{E})$. Since D' is also closed, it follows that $D' = C(\widehat{E})$. Hence,

$$C(\widehat{D}) \cong D' = C(\widehat{E})$$

Hence, $\widehat{D} = \widehat{E}$, which in turn implies that $D = E$. If b is not necessarily self-adjoint, note that b^*b is invertible in A since b is invertible in A. Then

$$(b^*b)^{-1}b^*b = e,$$

and

$$b^{-1} = (b^*b)^{-1}b^*.$$

By the argument above, $(b^*b)^{-1} \in B$. Hence, $b^{-1} \in B$. □

Proof. (Proposition 13.1) We first show existence. Let $B = C^*(a, e)$. Since B is a unital commutative C^*-algebra, the Gelfand transform gives us an isometric $*$-isomorphism

$$\Psi : B \to C(\widehat{B}).$$

The key argument is to explicitly identity $\widehat{B}$ for $B = C^*(a, e)$. Note that any non-zero, linear and multiplicative map on B is uniquely defined by its action on a. Hence, the map

$$\tau : \widehat{B} \to \sigma_A(a)$$
$$\omega \mapsto \omega(a)$$

is a continuous bijection. Since $\widehat{B}$ and $\sigma_A(a)$ are compact and Hausdorff, τ is a homeomorphism. Then we get an isometric *-isomorphism

$$\Theta : C(\sigma_A(a)) \to C(\widehat{B})$$

by

$$\Theta(f)(\omega) = f(\tau(\omega)) = f(\omega(a))$$

for $f \in C(\sigma_A(a))$, $\omega \in \widehat{B}$. The desired conclusion follows by letting $\Phi = \Psi^{-1} \circ \Theta$. Indeed, note that $\Theta(\mathrm{Id}_{\sigma_A(a)})(\omega) = \tau(\omega) = \omega(a)$. Therefore, $\Theta(\mathrm{Id}_{\sigma_A(a)}) = \mathrm{Ev}_a$. Hence, $\Psi^{-1} \circ \Theta(\mathrm{Id}_{\sigma_A(a)}) = a$ as required. We now show uniqueness. Since $\Phi_a(1_{\sigma_A(a)})$ and $\Phi_a(\mathrm{Id}_{\sigma_A(a)})$ are fixed, Φ_a is already uniquely defined for all Laurent polynomials since Φ_a is a *-homomorphism. These polynomials form a dense subalgebra of $C(\sigma_A(a))$ by the Stone-Weierstrass theorem. Thus Φ_a is unique. □

Proposition 13.1 shows that continuous functional calculus can be used to reduce some abstract problems involving normal elements of a (not necessarily commutative!) C^*-algebra to problems about function algebras.

Remark 13.3. *In what follows, we shall write $f(a)$ for $\Psi_a(f)$ from time to time.*

Corollary 13.4. *Let A be a unital C^*-algebra A, and let a be a normal element of A. Let $f \in C(\sigma_A(a))$. We have the following:*

(1) *$f(a) \in A$ is normal*
(2) ***(Spectral Mapping Theorem)*** *$f(\sigma_A(a)) = \sigma_A(f(a))$*

Proof. The proof is given below:

(1) We have,

$$f(a)^*f(a) = \Phi_a(f)^*\Phi_a(f) = \Phi_a(\overline{f}f) = \Phi_a(f\overline{f}) = \Phi_a(f)\Phi_a(f)^* = f(a)f(a)^*$$

(2) Since $f(a) \in C^*(a, e)$ and Ψ_a is a $*$-isometric isomorphism, we have

$$\sigma_A(f(a)) := \sigma_A(\Phi_a(f)) = \sigma_A(f) = f(\sigma_A(a)).$$

The last equality follow since the spectrum of f is simply its range.

This completes the proof. □

We now present some applications of Proposition 13.1 and Corollary 13.4. We first present a complete characterization of self-adjoint, unitary and projection elements in a unital C^*-algebra.

Corollary 13.5. *Let A be a unital C^*-algebra, and let $a \in A$ be a normal element. Then:*

(1) *a is self-adjoint if and only if $\sigma_A(a) \subseteq \mathbb{R}$.*
(2) *a is unitary if and only if $\sigma_A(a) \subseteq \mathbb{S}^1$.*
(3) *a is a projection if and only if $\sigma_A(a) \subseteq \{0, 1\}$.*

Proof. The proof is given below:

(1) If a is self-adjoint, we first show that e^{ia} is unitary. Since the $*$ operation is isometric and hence continuous, we have

$$(e^{ia})^* = \left(\sum_{k=0}^{\infty} \frac{i^k a^k}{k!}\right)^* = \sum_{k=0}^{\infty} \frac{(-i)^k a^k}{k!} = e^{-ia} = (e^{ia})^{-1}$$

This shows that e^{ia} is unitary. By Proposition 10.5(5), we have $\sigma_A(e^{ia}) \subseteq \mathbb{S}^1$. Let $\lambda \in \sigma_A(a)$. Define

$$b := \sum_{n=1}^{\infty} \frac{i^n}{n!}(a - \lambda)^{n-1}.$$

Note that b commutes with a. We have

$$e^{ia} - e^{i\lambda} e = \left(e^{i(a-\lambda)} - e\right) e^{i\lambda} = (a - \lambda e) b e^{i\lambda}.$$

Since b commutes with a, and hence with $(a-\lambda e)$, and since $(a-\lambda e)$ is not invertible, we conclude that $e^{ia} - e^{i\lambda}$ is not invertible[10]. Therefore, $e^{i\lambda} \in \sigma_A(e^{ia}) \subseteq \mathbb{S}^1$. So we must have $\lambda \in \mathbb{R}$. Conversely, assume that $\sigma_A(a) \in \mathbb{R}$. Then,

$$a^* = \Phi_a(\mathrm{Id}_{\sigma_A(a)})^* = \Phi_a(\overline{\mathrm{Id}_{\sigma_A(a)}}) = \Phi_a(\mathrm{Id}_{\sigma_A(a)}) = a.$$

Hence, a is self-adjoint.

(2) The forward direction was proved in Proposition 10.5(5). Conversely, assume that $\sigma_A(a) \subseteq \mathbb{S}^1$. Then,

$$a^* a = \Phi_a(\mathrm{Id}_{\sigma_A(a)})^* \Phi_a(\mathrm{Id}_{\sigma_A(a)}) = \Phi_a(\overline{\mathrm{Id}_{\sigma_A(a)}}\, \mathrm{Id}_{\sigma_A(a)}) = \Phi_a(1_{\sigma_A(a)}) = e$$

Similarly, $aa^* = e$. Hence, a is a unitary.

(3) If a is a projection, then $a^2 = a$. By Corollary 13.4(2), we have

$$\{0\} = \sigma_A(0) = \sigma_A(a^2 - a) = \{\lambda^2 - \lambda \mid \lambda \in \sigma(A)\}$$

This shows that $\sigma_A(A) \subseteq \{0, 1\}$. The converse follows as in (2).

This completes the proof. □

Here is another application:

[10] Here we have used the fact that if $xy = yx$ and xy is invertible, then x and y are invertible. This statement is true in any ring with a unit.

Corollary 13.6. *Let A be a unital C^*-algebra. If $a \in A$ is unitary such that $\sigma_A(a) \subsetneq \mathbb{S}^1$, then there exists a self-adjoint $b \in A$ such that $a = e^{ib}$.*

Proof. WLOG, we can assume that $-1 \notin \sigma_A(a)$. Let $\ln : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ denote the principal branch of the logarithm function. Note that we have $e^{\ln(\cdot)} = \mathrm{Id}_{\sigma_A(a)}(\cdot)$. Since $|\mathrm{Id}_{\sigma_A(a)}| = 1$, the real part of ln restricted to $\sigma_A(a)$ vanishes. Hence, $\ln|_{\sigma_A(a)} = ih$ for some real-valued function $h \in C(\sigma_A(a))$. Let $b = \Phi_a(h)$. Since h is real-valued, b is self-adjoint. Moreover, we have

$$a = \Phi_a(\mathrm{Id}_{\sigma_A(a)}) = \Phi_a(e^{ih}) = e^{i\Phi_a(h)} = e^{ib}.$$

This completes the proof. □

Here is another application:

Corollary 13.7. *Let A, B be unital C^*-algebras and let $\varphi : A \to B$ be a unital $*$-homomorphism. If $a \in A$ is normal element, then for every $f \in C(\sigma_A(a))$, we have $\varphi(f(a)) = f(\varphi(a))$.*

Proof. Note that $\varphi(a)$ is normal and $\sigma_B(\varphi(a)) \subseteq \sigma_A(a)$, and so the restriction of f to $\sigma_B(a)$ is continuous. Define two unital $*$-homomorphisms $\Phi_1, \Phi_2 : C(\sigma_A(a)) \to B$ by

$$\Phi_1(f) := \varphi(\Phi_a(f)) = \varphi(f(a)),$$

and

$$\Phi_2(f) := \Phi_{\varphi(a)}(f|_{\sigma_B(\varphi(a))}) = f|_{\sigma_B(\varphi(a))}(\varphi(a)).$$

It is easy to check that they both map $1_{\sigma_A(a)}$ and $\mathrm{Id}_{\sigma_A(a)}$ to 1_B and $\varphi(a)$, respectively. Thus they agree on all polynomials of two variables z and $\overline{z}$ over $\sigma_A(a)$ and since they are continuous, they agree on all of $C(\sigma_A(a))$. □

14. Positive Elements

Continuous Functional Calculus (CFC) is a powerful tool for manipulating normal elements of a C^*-algebra. Granted, every element of a non-commutative C^*-algebra is not normal. Nonetheless, we can associate a self-adjoint element for every element in a non-commutative C^*-algebra, allowing us to spread the influence of the functional calculus to an entire non-commutative C^*-algebra. In this section, we discuss how positive elements can be defined via CFC.

Definition 14.1. Let A be a C^*-algebra. A self-adjoint element $a \in A$ is **positive** if $\sigma_A(a) \subseteq \mathbb{R}^+$.

Remark 14.2. *If A is a C^*-algebra, the subset of positive elements is denoted by A_+, and if $a \in A_+$ we write $a \geq 0$.*

Example 14.3. Let $A = C_0(X)$ for some locally compact topological space. Positive elements in are non-negative real-valued functions.

We now discuss applications of CFC to produce new elements in a C^*-algebra.

Proposition 14.4. *Let A be a unital C^*-algebra.*

(1) *If $a \in A$ is a self-adjoint element, then a can be written uniquely as*

$$a = a_+ - a_- \qquad a_\mp a_\pm = 0$$

for $a_\pm \in A_+$. The elements $a_\pm$ are called the positive/negative parts of a.

(2) *Every $a \in A_+$ and $n \geq 1$, there exists a unique element $b \in A_+$ such that $a = b^n$. The element b is called the n-th root of a.*

Proof. The proof is given below:

(1) Consider the functions
$$f^+(t) = \max(t, 0), \qquad f^-(t) = -\min(t, 0)$$
We have that $f^{\pm} \in C(\sigma_A(a))$ and $f^+(t) - f^-(t) = t$[11]. By CFC, we have that,
$$a = a_+ - a_-$$
such that $a_{\pm} = \Psi_a(f^{\pm})$. Since $f^{\pm}$ are non-negative real-valued functions, we have that $\sigma_A(a_{\pm}) \subseteq \mathbb{R}^+$ by the spectral mapping theorem, Hence, $a_{\pm}$ are positive elements. Moreover, note that $f^{\mp} \cdot f^{\pm} = 0$. Hence, $a^{\mp}a^{\pm} = 0$ by CFC.

(2) For each $n \in \mathbb{N}$, consider the function
$$f_n : \mathbb{R}^+ \to \mathbb{R}^+, x \mapsto \sqrt[n]{x},$$
which is a continuous function on $\sigma_A(a) \subseteq \mathbb{R}^+$. Define, $b = \Phi_a(f_n)$. Then
$$b^n = \Phi_a(f_n)^n = \Phi_a(f_n^n) = \Phi_a(\mathrm{Id}_{\sigma_A(a)}) = a$$
by CFC. By the spectral mapping theorem. we have
$$\sigma_A(b) = \sigma_A(f_n(a)) = f_n(\sigma_A(a)) \subseteq \mathbb{R}^+$$
i.e., b is positive. If $c \in A_+$ is another positive element such that $c^n = a = b^n$, then
$$c = \Phi_a(f_n) = \Phi_a(f_n) = b$$
by uniqueness of the inverse of the square root function defined on $\sigma_A(a)$. This proves uniqueness.

This completes the proof. □

If X is compact and $f \in C(X)_+$, then notice that $|f(x) - t| \leq t$ for every real number $t \geq \|f\|$. Conversely, if $|f(x) - t| \leq t$ for some $t \geq \|f\|$, then $f(x) \geq 0$ for all x and so $f \geq 0$. These observations are behind some of the statements in the next result.

Lemma 14.5. *Let A be a unital C^*-algebra, and let a be a self-adjoint element. Then the following are equivalent.*

(1) $a \geq 0$
(2) $\|\alpha e - a\| \leq \alpha$ *for all* $\alpha \geq \|a\|$
(3) $\|\alpha e - a\| \leq \alpha$ *for some* $\alpha \geq \|a\|$.

Proof. We first prove (1) implies (2). Since $a \geq 0$, $a = a^*$. Hence, $C^*(a)$ is abelian. Recall that $C(\sigma_A(a)) \cong C^*(a)$, such that the identity function in $C(\sigma_A(a))$ corresponds to a. Since $a \geq 0$, we have that the identity function is in $C(\sigma_A(a))_+$. Since $\sigma_A(a)$ is compact, the discussion preceding the statement of the proposition then implies that (2) is true if we take f to be the identity function. Clearly, (2) implies (3). (3) implies (1) follows from an argument similar to that that implies (1) implies (2). □

Lemma 14.5 gives us a nice characterization of positive elements. We have the following corollaries.

Corollary 14.6. *Let A be a unital C^*-algebra.*

[11]Since a is self-adjoint, we can assume that $t \in \mathbb{R}$.

(1) *Then* A_+ *is closed.*
(2) *If* $a, b \in A_+$, *then* $a + b \in A_+$.
(3) *If* $a, b \in A_+$ *and* a *and* b *commute, then* $ab \in A_+$.
(4) a *is positive if and only if* $a = b^*b$ *for some* $b \in A$.

Proof. The proof is given below:

(1) Suppose $(a_n) \in A_+$ converges to $a \in A$. Then
$$\|a_n^* - a^*\| = \|a_n - a\| \to 0,$$
and so $(a_n^*) = (a_n) \in A_+$ converges to a^*. Hence $a^* = a$. Moreover, we have that $(\|a_n\|)$ converges to $\|a\|$. By Lemma 14.5
$$\|\|a_n\|e - a_n\| \leq \|a_n\|$$
for each $n \in \mathbb{N}$. Hence,
$$\|\|a\|e - a\| \leq \|a\|$$
for each $n \in \mathbb{N}$. By Lemma 14.5, we have that $a \geq 0$.

(2) Clearly, $a + b$ is self-adjoint. It suffices to assume that $\|a\|, \|b\| \leq 1$[12]. But
$$\|1 - \frac{1}{2}(a + b)\| = \frac{1}{2}\|(1 - a) + (1 - b)\| \leq 1$$
by Lemma 14.5. Hence, $\frac{1}{2}(a + b) \geq 0$ by Lemma 14.5, which implies that $a + b \geq 0$.

(3) Note that,
$$(a + b)^2 = a^2 + 2ab + b^2$$
By the spectral mapping theorem and (2) above, $a^2, b^2, (a + b)^2 \in A_+$. Hence, $2ab \in A_+$ which in turn implies that $ab \in A_+$.

(4) The forward implication follows from Proposition 14.4(2) by simply taking the square root of a. Conversely, clearly a is self-adjoint. We show that $a = b^*b$ implies that $\sigma_A(a) \subseteq \mathbb{R}^+$. Note that $a = b^*b$. Hence, we shall apply CFC to $a = b^*b$. Define
$$f(t) = \begin{cases} \sqrt{t} & \text{if } t \geq 0, \\ 0 & \text{otherwise} \end{cases} \qquad g(t) = \begin{cases} 0 & \text{if } t \geq 0, \\ \sqrt{-t} & \text{otherwise.} \end{cases}$$
Then for all $t \in \mathbb{R}$, we have $f(t)g(t) = 0$ and $f(t)^2 - g(t)^2 = t$. Since f and g both vanish at 0, we get self-adjoint elements of $u = \Phi_{b^*b}(f)$ and $v = \Phi_{b^*b}(g)$ of A such that
$$\Phi_{b^*b}(g)\Phi_{b^*b}(f) = \Phi_{b^*b}(g)\Phi_{b^*b}(f) = 0, \quad \Phi_{b^*b}(f)^2 - \Phi_{b^*b}(g)^2 = b^*b$$
But then
$$\Phi_{b^*b}(g)(\Phi_{b^*b}(f)^2 - \Phi_{b^*b}(g)^2)\Phi_{b^*b}(g) = -\Phi_{b^*b}(g)^4.$$
Thus[13]
$$\sigma_A((bv)^*bv) = \sigma_A(-\Phi_{b^*b}(g)^4) \subseteq (-\infty, 0].$$
Thus $-\Phi_{b^*b}(g)^4 = 0$. Since v is self-adjoint, this means $\Phi_{b^*b}(g) = 0$. But then $b^*b = \Phi^2_{b^*b}(f)$, and $\Phi^2_{b^*b}(f)$ is positive by the spectral mapping theorem for normal elements.

This completes the proof. □

[12]This because is a positive-scalar of a positive element is a positive element

[13]Here we use the result that if $a \in A$ and $\sigma_A(a^*a) \subseteq (-\infty, 0]$, then $a = 0$.

Remark 14.7. *Note that Corollary 14.6(3) is not true in general. Indeed, let*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$A, B \in M_2(\mathbb{C})$ are positive. However, AB is not positive since

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is not even self-adjoint.

Remark 14.8. *All claims about positive elements are true in a non-unital C^*-algebra. These facts can be proved by passing to the unitization of a non-unital C^*-algebra.*

15. States

We discuss states in this section. These will be useful in the GNS construction to be discussed later on.

Definition 15.1. Let A be a C^*-algebra. A **state** on A is a linear functional such that:

1. φ is positive. That is, $\varphi(A_+) \subseteq \mathbb{R}^+$. Equivalently, $\varphi(a^*a) \geq 0$ for each $a \in A$
2. The norm of φ is one. That is,
$$\|\varphi\| := \sup\{|\varphi(a)| : \|a\| = 1\} = 1.$$

The subset $S(A) \subseteq A^*_{\leq 1}$ consisting of states is called the **state space**.

Proposition 15.2. *Let A be a C^*-algebra, and let φ be a positive linear functional on A.*

1. *φ is $*$-preserving. That is,*
$$\varphi(a^*) = \overline{\varphi(a)}$$
2. *(**Cauchy-Schwartz**) For $a, b \in A$, we have,*
$$|\varphi(b^*a)|^2 \leq \varphi(b^*b)\varphi(a^*a)$$
such that we have equality if and only if $\varphi(a^*b) = \varphi(b^*a)$.
3. *If A is unital, $\varphi(e) = \|\varphi\|$.*

Proof. The proof is given below:

1. Note that Proposition 10.5 that it suffices to show that φ maps self-adjoint elements to self-adjoint elements. If $a \in A$ is a self-adjoint element, then $a = a_+ - a_-$ for some $a_\pm \in A_+$. Note that
$$\varphi(a) = \varphi(a_+) - \varphi(a_-) \in \mathbb{R},$$
since φ is a positive linear functional. The claim follows.
2. Let $\lambda \in \mathbb{C}$. Since φ is positive, it follows that
$$\varphi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\varphi(a^*a) + \overline{\lambda}\varphi(a^*b) + \lambda\varphi(b^*a) + \varphi(b^*b) \geq 0.$$
Because this expression must be real for all $\lambda \in \mathbb{C}$, it follows that $\overline{\varphi(a^*b)} = \varphi(b^*a)$. Hence, we get the inequality,
$$|\lambda|^2\varphi(a^*a) + 2\mathrm{Re}(\lambda\varphi(b^*a)) + \varphi(b^*b) \geq 0.$$
Let $\gamma \in \mathbb{S}^1$ such that $\gamma\varphi(b^*a) = |\varphi(b^*a)|$. Given $t \in \mathbb{R}$, put $\lambda = t\gamma$. Hence, we get the inequality,
$$t^2\varphi(a^*a) + 2t|\varphi(b^*a)| + \varphi(b^*b) \geq 0.$$

As we can do this for any $t \in \mathbb{R}$ and this is a real quadratic, for this to be always non-negative we need $b^2 \leq 4ac$, i.e.,

$$4|\varphi(b^*a)|^2 \leq 4\varphi(a^*a)\varphi(b * b)$$

The desired result now follows.

(3) Clearly, $\varphi(e) \leq \|\varphi\|$. Using $b = e$ in (2), we have

$$|\varphi(a)|^2 \leq \varphi(a^*a)\varphi(e) \leq \|\varphi\|\|a^*a\|\varphi(e) = \|\varphi\|\|a\|^2\varphi(e)$$

Taking the supremum over all a of norm one, we get

$$\|\varphi\|^2 \leq \|\varphi\|\varphi(e)$$

It follows that

$$\|\varphi\| \leq \varphi(e).$$

Hence, $\varphi(e) = \|\varphi\|$.

This completes the proof. □

16. Representations of C^*-Algebras

Definition 16.1. Let A be a C^*-algebra. A **representation** of A is a pair, $(\pi, \mathscr{H})$, where $\mathscr{H}$ is a Hilbert space and $\pi : A \to \mathscr{B}(\mathscr{H})$ is a $*$-homomorphism.

Definition 16.2. Let A be a C^*-algebra. Two representations of A, $(\pi_1, \mathscr{H}_1)$, $(\pi_2, \mathscr{H}_2)$, are **unitarily equivalent** if there is a unitary operator $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$U\pi_1(a) = \pi_2(a)U,$$

for all a in A. In this case, we write $(\pi_1, \mathscr{H}_1) \sim_U (\pi_2, \mathscr{H}_2)$ or $\pi_1 \sim_U \pi_2$.

Remark 16.3. *We usually write that π is a representation of A on $\mathscr{H}$.*

An important problem is representation theory is to understand irreducible representations of an abstract mathematical object.

Definition 16.4. Let A be a C^*-algebra and let $(\pi, \mathscr{H})$ be a representation of A. A subspace $\mathscr{N} \subseteq \mathscr{H}$ is said to be **invariant** if $\pi(a)\mathscr{N} \subseteq \mathscr{N}$, for all a in A. The representation $(\pi, \mathscr{H})$ is said to be **irreducible** if the only invariant subspaces of $\{0\}$ and $\mathscr{H}$.

An important technique in representation theory is to construct 'larger' representations from 'smaller' representations by means of algebraic operation.

Definition 16.5. Let A be a C^*-algebra and $(\pi_\iota, \mathscr{H}_\iota)$, $\iota \in I$, be a collection of representations of A. The **direct sum representation** is $(\bigoplus_{\iota \in I} \pi_\iota, \bigoplus_{\iota \in I} \mathscr{H}_\iota)$, where $\bigoplus_{\iota \in I} \mathscr{H}_\iota$ consists of tuples, $x = (x_\iota)_{\iota \in I}$ satisfying $\sum_{\iota \in I} \|x_\iota\|^2 < \infty$ and

$$(\bigoplus_{\iota \in I} \pi_\iota(a)x)_\iota = \pi_\iota(a)x_\iota, \quad \iota \in I.$$

Proposition 16.6. *Let A be a C^*-algebra, and let $(\pi, \mathscr{H})$ be a representation of A. A closed subspace $\mathscr{N} \subseteq \mathscr{H}$ is invariant if and only if $\mathscr{N}^\perp$ is invariant.*

Proof. Assume that $\mathscr{N} \subseteq \mathscr{H}$ is a closed invariant subspace. Let x in $\mathscr{N}^\perp$ $y \in N$ and $a \in A$. We have

$$\langle \pi(a)x, y\rangle = \langle x, \pi(a)^*y\rangle = \langle x, \pi(a^*)y\rangle = 0,$$

since $\pi(a^*)\mathscr{N} \subseteq \mathscr{N}$. Hence, $\pi(a)x\mathscr{N}^\perp$, showing that $\mathscr{N}^\perp$ is invariant. The converse follows from the observation that $(\mathscr{N}^\perp)^\perp = \mathscr{N}$. □

The previous proposition shows that it is possible to define two representations of A by simply restricting the operators to either $\mathscr{N}$ or $\mathscr{N}^\perp$. Moreover, the direct sum of these two representations is unitarily equivalent to the original representation. That is, we have

$$(\pi, \mathscr{H}) \sim_U (\pi|_{\mathscr{N}}, \mathscr{N}) \oplus (\pi|_{\mathscr{N}^\perp}, \mathscr{N}^\perp).$$

We now discuss a characterizion irreducible representations of a C^*-algebra.

Definition 16.7. Let A be a C^*-algebra, and let $(\pi, \mathscr{H})$ be a representation of A. A vector $x \in \mathscr{H}$ is **cyclic** if

$$\overline{\operatorname{span}\{\pi(a)x \mid a \in A\}} = \mathscr{H}.$$

Remark 16.8. *We say that the representation is a cyclic representation if each non-zero vector in $\mathscr{H}$ is a cyclic vector.*

Remark 16.9. *A representation, $(\pi, \mathscr{H})$ of a C^*-algebra A is non-degenerate if the only $x \in \mathscr{H}$ such that $\pi(a)x = 0$ for all a in A is $x = 0$. Otherwise, the representation is degenerate. It can be easily showed that a representation $(\pi, \mathscr{H})$ of a unital C^*-algebra is non-degenerate if and only if $\pi(a) = I_{\mathscr{H}}$ implies that $a = e$.*

Proposition 16.10. *Let A be a C^*-algebra, and let $(\pi, \mathscr{H})$ be a non-degenerate representation of A. $(\pi, \mathscr{H})$ is an irreducible representation if and only if the representation $(\pi, \mathscr{H})$ is cyclic.*

Proof. Assume that $(\pi, \mathscr{H})$ is irreducible. Let x be a non-zero vector, then

$$\operatorname{span}\{\pi(a)x \mid a \in A\}$$

is an invariant subspace and its closure is a closed invariant subspace. If it is 0, then the representation is degenerate, which is impossible. Otherwise, it must be $\mathscr{H}$, meaning that x is a cyclic vector for π. Conversely, suppose that $(\pi, \mathscr{H})$ is non-degenerate, but reducible. Let $\mathscr{N}$ be a proper closed invariant subspace which is neither 0 nor $\mathscr{H}$. If x is any non-zero vector in $\mathscr{N}$, then

$$\operatorname{Span}\{\pi(a)x \mid a \in A\} \subseteq \mathscr{N}$$

Hence, it cannot be dense in $\mathscr{H}$. This shows that $(\pi, \mathscr{H})$ is not a cyclic representation. □

We end this section give a more useful criterion for a representation to be reducible.

Proposition 16.11. *(**Schur's Lemma**) Let A be a C^*-algebra, and let $(\pi, \mathscr{H})$ be a non-degenerate representation of A. Then $(\pi, \mathscr{H})$ is irreducible if and only if the only positive operators which commute with its image are scalar multiplies of the identity operator.*

Proof. First assume that $(\pi, \mathscr{H})$ is a reducible representation. Let $\mathscr{N} \subseteq \mathscr{H}$ be a non-trivial proper closed invariant subspace of $\mathscr{H}$. Let p be the orthogonal projection onto $\mathscr{N}$. We have that $p = p^*$ and $\sigma_A(p) \in \{0, 1\}$, which means that p is positive. We check that it commutes with $\pi(a)$, for any a in A. If $x \in \mathscr{N}$, we know that $\pi(a)x \in \mathscr{N}$ and so

$$(p\pi(a))x = p(\pi(a)x) = \pi(a)x = \pi(a)(px) = (\pi(a)p)x.$$

On the other hand, if x is in $\mathscr{N}^\perp$, then so is $\pi(a)x$ and

$$(p\pi(a))x = p(\pi(a)x) = 0 = \pi(a)(0) = \pi(a)(px) = (\pi(a)p)x.$$

Since every vector in $\mathscr{H}$ is the sum of two as above, we see that

$$p\pi(a) = \pi(a)p$$

for each $a \in A$. As both $\mathscr{N}$ and $\mathscr{N}^\perp$ are non-empty, this operator is not a scalar multiple of the identity operator on $\mathscr{H}$. Conversely, suppose that p is a positive that is not a scalar multiple of the identity operator on $\mathscr{H}$, but commutes with every element of $\pi(a)$ for each $a \in A$. We must have that

$$\sigma_A(p) = \{0, 1\}$$

We may then find non-zero continuous functions[14] f, g on $\sigma_A(p)$ whose product is zero. By CFC, $f(p)$ and $g(p)$ are well-defined operators in $\mathscr{B}(\mathscr{H})$. Since f is non-zero, the operator $\Phi_p(f)$ is non-zero. Let $\mathscr{N}$ denote the closure of its range, which is a non-zero subspace of $\mathscr{H}$. On the other hand, $\Phi_p(g)$ is also a non-zero operator, but it is zero on the range of $\Phi_p(f)$ and hence on N. This implies that $\mathscr{N}$ is a proper subspace of H. Note that $\Phi_a(f)$ commutes with $\pi(a)$ for each $a \in A$. Let $a \in A$. Indeed, for any $\epsilon > 0$, we may find a polynomial $q(x) \in C(\sigma_A(p))$ such that

$$\|f - q\|_\infty < \epsilon$$

in $C(\sigma_A(p))$, This means that

$$\|\Phi_p(f) - \Phi_p(q)\| < \epsilon$$

It is clear that $\Phi_p(q)$ will commute with $\pi(a)$, since p commutes with $\pi(a)$. A triangle inequality type argument now shows tht $\Phi_p(f)$ must also commute with $\pi(a)$. Finally, we claim that $\mathscr{N}$ is invariant under $\pi(a)$. In fact, it suffices to check that the range of $\Phi_p(f)$ is invariant. But if $x \in \mathscr{H}$, we have

$$\pi(a)(\Phi_p(f)x) = \pi(a)\Phi_p(f)x = \Phi_p(f)\pi(a)x \in \Phi_p(f)\mathscr{H},$$

Hence, $\mathscr{N}$ is a proper, closed invariant subspace of $\mathscr{H}$. This completes the proof. □

17. Gelfand-Naimark-Segal Construction

In this section, we explicitly construct a representation of a given C^*-algebra. The basic idea is that multiplication allows one to see the elements of a C^*-algebra acting as linear transformations of itself. The problem is, of course, that the C^*-algebra does not usually have the structure of a Hilbert space. To produce an inner product or bilinear form, we use the linear functionals on the C^*-algebra in a clever way, leading to the Gelfand-Naimark-Segal (GNS) construction. First consider the following example.

Example 17.1. Let A be a unital C^*-algebra, and $(\pi, \mathscr{B}(\mathscr{H}))$ be a non-degenerate representation of A. Let $x \in \mathscr{H}$ such that $\|x\| = 1$, and consider the linear functional

$$\varphi(a) := \langle \pi(a)x, x \rangle$$

on A. Note that,

$$\begin{aligned}
\varphi(a^*a) &= \langle \pi(a^*a)x, x \rangle \\
&= \langle \pi(a^*)\pi(a)x, x \rangle \\
&= \langle \pi(a)^\dagger \pi(a)x, x \rangle \\
&= \langle \pi(a)x, \pi(a)x \rangle \\
&\geq 0
\end{aligned}$$

[14]Use Uryshon's lemma or simply consider bump functions.

Here † denote the Hilbert space adjoint operator. Moreover, note that,

$$\begin{aligned}\|\varphi(a)\| &= |\langle \pi(a)x, x\rangle| \\ &\leq \|\pi(a)x\|_{\mathscr{H}}\|x\|_{\mathscr{H}} \\ &\leq \|\pi(a)\|_{\mathscr{B}(\mathscr{H})}\|x\|_{\mathscr{H}}\|x\|_{\mathscr{H}} \\ &= \|\pi(a)\|_{\mathscr{B}(\mathscr{H})}\|x\|_{\mathscr{H}}^2 \\ &\leq \|a\|_A\|x\|_{\mathscr{H}}^2 = \|a\|_A\end{aligned}$$

The last inequality follows since π is a $*$-algebra which are known to be contractive. Hence, $\|\varphi\| \leq 1$. But note that,

$$\varphi(e) = \langle \pi(e)x, x\rangle = \langle I_{\mathscr{H}}x, x\rangle = \langle x, x\rangle = 1.$$

Hence, $\|\varphi\| = 1$. This shows that φ is a state.

Example 17.1 implies that we can associate a state corresponding to a non-degenerate representation of a C^*-algebra. We now show that the converse is true as well, which is the GNS construction.

Proposition 17.2. *(**GNS Construction**) Let A be a C^*-algebra. If φ is any state on A, there is a non-degenerate representation $(\pi_\varphi, \mathscr{H}_\varphi)$ and a unit vector $x_\varphi \in \mathscr{H}_\varphi$ such that*

$$\varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi\rangle_\varphi$$

for any $a \in A$. Moreover, the representation is unique up to unitary equivalence. That is, if $\pi : A \to \mathscr{B}(\mathscr{H})$ is another representation with cyclic unit vector $x \in \mathscr{H}$ satisfying

$$\varphi(a) = \langle \pi x, x\rangle$$

for all $a \in A$, then there exists a unique unitary $U : \mathscr{H}_\varphi \to \mathscr{H}$ such that

$$Ux_\varphi = x$$

and

$$U\pi_\varphi(a) = \pi(a)U$$

for all $a \in A$.

Proof. If φ is a state, then φ defines a sesqui-linear form. Define the set,

$$\mathscr{N}_\varphi = \{a \in A \mid \varphi(a^*a) = 0\}$$

We first show that φ is a closed left ideal. Clearly, φ is closed since φ is continuous[15]. Moreover, $\mathscr{N}_\varphi$ is a vector subspace. Indeed, if $a, b \in \mathscr{N}_\varphi$, and $\lambda \in \mathbb{C}$, then

$$\varphi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\varphi(a^*a) + \lambda\varphi(b^*a) + \overline{\lambda}\varphi(a^*b) + \varphi(b^*b) = 0$$

The first and fourth terms are zero by assumption. The second and third terms are zero by Proposition 15.2(2). If $a \in \mathscr{N}_\varphi$ and $b \in A$, consider the functional

$$\psi(c) = \varphi(a^*ca)$$

for any $c \in A$. This is clearly another positive linear functional, and so we have

$$\|\psi\| = \psi(1) = \varphi(a^*a) \tag{*}$$

Then we have

$$0 \leq \varphi((ba)^*ba) = \varphi(a^*b^*ba) = \psi(b^*b) \leq \|\psi\|\|b^*b\| = \varphi(a^*a)\|b\|^2 = 0$$

[15]Note that φ is the composition of the maps $a \mapsto (a, a^*) \mapsto \varphi(a^*a)$ which is indeed continuous.

This shows that $ba \in \mathscr{N}_\varphi$. Hence, $\mathscr{N}_\varphi$ is a closed left ideal. Consider the map,

$$\begin{aligned}(,) : A \times A &\to \mathbb{C}\\ (a,b) &\mapsto \varphi(a^*b)\end{aligned}$$

Our discussion above shows that the map defined above is a sesquilinear form. Indeed, it is easy to check that the map is linear in the second argument. Moreover, it is conjugate symmetric since,

$$(a,b) = \varphi(a^*b) = \varphi((ba^*)^*) = \overline{\varphi(ba^*)} = \overline{(b,a)}$$

But it might not be an inner product since $\mathscr{N}_\varphi \neq \emptyset$. Consider $V_\varphi = A/\mathscr{N}_\varphi$. We can now define an *honest* inner product on V_φ by the formula:

$$\langle a + \mathscr{N}_\varphi, b + \mathscr{N}_\varphi \rangle_\varphi := \varphi(b^*a).$$

To see that this is well-defined, note that, for any $x, y \in \mathscr{N}_\varphi$ and $a, b \in A$,

$$\begin{aligned}\varphi((b+y)^*(a+x)) &= \varphi(b^*a + b^*x + y^*a + y^*x)\\ &= \varphi(b^*a) + \varphi(b^*x) + \varphi(y^*a) + \varphi(y^*x)\\ &= \varphi(b^*a).\end{aligned}$$

The last equality follows since $\mathscr{N}_\varphi$ is a left ideal. Let $\mathscr{H}_\varphi$ to be the completion of V_φ with respect to the norm induced by $\langle \cdot, \cdot \rangle_\varphi$. The action of A on A by left multiplication induces a representation $\pi_\varphi : A \to \mathscr{B}(\mathscr{H}_\varphi)$ is given by left multiplication:

$$\pi_\varphi(a)(b + \mathscr{N}_\varphi) = ab + \mathscr{N}_\varphi.$$

The fact that $\mathscr{N}_\varphi$ is an ideal ensures that $\pi_\varphi(a)$ is a well-defined map from V_φ to V_φ for each $a \in A$. Moreover, the map $\pi_\varphi(a)$ is bounded from V_φ to V_φ for each $a \in A$. Indeed,

$$\begin{aligned}\|\pi_\varphi(a)\|^2 &= \sup\{\langle \pi_\varphi(a)(x + \mathscr{N}_\varphi), \pi_\varphi(a)(x + \mathscr{N}_\varphi)\rangle_\varphi \mid x \in A, \varphi(x^*x) = 1\}\\ &= \sup\{\varphi((ax)^*(ax)) \mid x \in A, \varphi(x^*x) = 1\}\\ &= \sup\{\varphi(x^*a^*ax) \mid x \in A, \varphi(x^*x) = 1\}\\ &\leq \sup\{\varphi(x^*x)\|a^*a\| \mid x \in A, \varphi(x^*x) = 1\}\\ &= \|a^*a\| = \|a\|^2.\end{aligned}$$

We have used the information in $(*)$. It is clear that $\pi_\varphi(a)$ is linear and multiplicative from V_φ to V_φ for each $a \in A$. We check that $\pi_\varphi(a)$ is $*$-preserving from V_φ to V_φ for each $a \in A$. For $a, b, c \in A$, we have

$$\begin{aligned}\langle \pi_\varphi(a^*)b + \mathscr{N}_\varphi, c + \mathscr{N}_\varphi\rangle_\varphi &= \langle a^*b + \mathscr{N}_\varphi, c + \mathscr{N}_\varphi\rangle_\varphi\\ &= \varphi(c^*(a^*b))\\ &= \varphi((ac)^*b)\\ &= \langle b + \mathscr{N}_\varphi, ac + \mathscr{N}_\varphi\rangle_\varphi\\ &= \langle b + \mathscr{N}_\varphi, \pi_\varphi(a)c + \mathscr{N}_\varphi\rangle_\varphi\\ &= \langle \pi_\varphi(a^*)b + \mathscr{N}_\varphi, c + \mathscr{N}_\varphi\rangle_\varphi.\end{aligned}$$

Since this holds for arbitrary b and $c \in A$, we conclude that $\pi_\varphi(a)^* = \pi_\varphi(a)^*$. A standard density argument now shows that the map $\pi_\varphi : A \to \mathscr{B}(\mathscr{H}_\varphi)$ is a well-defined, linear, multiplicative and $*$-preserving map. Consider the vector $x_\varphi = 1 + \mathscr{N}_\varphi$. Note that,

$$\begin{aligned} \|x_\varphi\| &= \|1 + \mathscr{N}_\varphi\| \\ &= \varphi(1^*1)^{1/2} \\ &= \varphi(1)^{1/2} \\ &= 1. \end{aligned}$$

If b is any element of A, it is clear that

$$\pi_\varphi(b)x_\varphi = b \cdot 1 + \mathscr{N}_\varphi = b + \mathscr{N}_\varphi.$$

It follows then that $\pi_\varphi(A)x_\varphi$ contains $A/\mathscr{N}_\varphi$ and is therefore dense in H_φ. Note that we have,

$$\langle \pi_\varphi(a)x_\varphi, x_\varphi\rangle_\varphi = \langle a + \mathscr{N}_\varphi, 1 + \mathscr{N}_\varphi\rangle = \varphi(a)$$

for each $a \in A$. This proves existence. Let $\pi : A \to \mathscr{B}(\mathscr{H})$ be another representation with cyclic unit vector $x \in \mathscr{H}$ satisfying

$$\varphi(a) = \langle \pi(a)x, x\rangle$$

for all $a \in A$. Let V denote the set

$$V = \{\pi(a)x : a \in A\} \subseteq \mathscr{H}.$$

Define

$$\begin{aligned} U : V_\varphi &\to V, \\ \pi_\varphi(a)x_\varphi &\mapsto \pi(a)x \end{aligned}$$

We first show that U is well-defined. Assume that $\pi_\varphi(a)x_\varphi = \pi_\varphi(b)x_\varphi$. We must show that $\pi(a)x = \pi(b)x$. Certainly, using that π is a $*$-morphism:

$$\begin{aligned} \langle \pi(a-b)x, \pi(a-b)x\rangle &= \langle \pi((a-b)^*(a-b))x, x\rangle \\ &= \varphi((a-b)^*(a-b)) \\ &= \langle \pi_\varphi((a-b)^*(a-b))x_\varphi, x_\varphi\rangle_\varphi \\ &= \langle \pi_\varphi(a-b)x_\varphi, \pi_\varphi(a-b)x_\varphi\rangle_\varphi \\ &= \|\pi_\varphi(a-b)x\|_\varphi^2 = 0. \end{aligned}$$

Hence, we have $\pi(a)x = \pi(b)x$. This shows that U is a well-defined map. We now check that U is an injective map. Assume that $\pi(a)x = \pi(b)x$ for some $a, b \in A$. Invoking the above calculation, we have,

$$\|\pi_\varphi(a-b)x\|_\varphi^2 = \|\pi(a-b)\|^2$$

Hence, $\pi_\varphi(a)x = \pi_\varphi(b)x$, implying that U is injective. By a simple inspection, U is surjective. Hence, U is a bijection. To see that U preserves the inner product, notice that for $a, b \in A$, the following holds:

$$\begin{aligned} \langle U(\pi_\varphi(a)x_\varphi), U(\pi(b)x_\varphi)\rangle &= \langle \pi(a)x, \pi(b)x\rangle \\ &= \langle \pi(b^*a)x, x\rangle \\ &= \varphi(b^*a). \end{aligned}$$

A verbatim argument shows that

$$\langle \pi_\varphi(a)x_\varphi, \pi(b)x_\varphi\rangle = \varphi(b^*a)$$

This implies that U preserves inner products. Since U preserves inner products, it follows that U is also bounded. Moreover, it is a simple matter to check that U is a linear map. Hence, we see that

$$U : V_\varphi \to V$$
$$\pi_\varphi(a)x_\varphi \mapsto \pi(a)x$$

is a is unitary map. Using the density of V_φ in $\mathscr{H}_\varphi$ (and that of V in $\mathscr{H}$), we may uniquely extend U to a bounded linear operator

$$\widetilde{U} : \mathscr{H}_\varphi \to \mathscr{H}$$

Simple density arguments can be used to show that $\widetilde{U}$ is surjective, preserves the inner product and is a unitary equivalence. The claim follows at once if we replace $\widetilde{U}$ by U. □

Remark 17.3. *A state, φ, is called a pure state if φ cannot be written as a non-trivial convex combination of states. With a bit more work, one can show that the representation constructed in Proposition 17.2 is irreducible if and only if φ is a pure state.*

18. Gelfand-Naimark-Segal Theorem

The study of C^*-algebras is motivated by the prime example of norm closed $*$-algebras of operators on Hilbert space. With this in mind, it is natural to find ways that a given abstract C^*-algebra may act as operators on Hilbert space. Such an object is called a representation of a C^*-algebra, and the study of representations of a C^*-algebra leads to the proof of the Gelfand-Naimark-Segal theorem:

Theorem 18.1. *(Gelfand, Naimark & Segal) Let A be a C^*-algebra. Then A is isometrically $*$-isomorphic to a $*$-closed subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space, $\mathscr{H}$.*

In order to prove Theorem 18.1, we will take the direct sum of *a lot* of the representations given in Proposition 17.2 to produce a faithful representation of a C^*-algebra. We must take into account one caveat, though. Proposition 17.2 is proved under the assumption that there exists a state defined on A. In order to take a direct sum of *a lot* of representations given in Proposition 17.2, we need to show the existence of *a lot* of states on A.

Lemma 18.2. *Let A be a unital C^*-algebra.*

(1) *If φ is a bounded linear functional on A which satisfies*

$$1 = \|\varphi\| = \varphi(e),$$

then φ is a state.

(2) *Let a be a non-zero, self-adjoint (hence normal) element of A. Then there is a state ψ on A such that*

$$|\psi(a)| = \|a\|.$$

Proof. The proof is given below:

(1) Skipped.

(2) Let $B = C^*(a, e) \subseteq A$. Let $\lambda = r(a)$[16]. Let $\mathrm{Ev}_\lambda : C(\sigma_A(a)) \to \mathbb{C}$ be given by evaluation at λ. Since Ev_λ is a character on $C(\sigma_A(a))$, it is, in particular, a state on $C(\sigma_A(a))$. Since $B \cong C^*(a, e)$, we have furnished a state on B. Since B is a closed subspace of A, the Hahn-Banach theorem allows us to extend it to a linear functional $\psi \in A^*$ with the same norm (i.e., $\|\psi\| = 1$). As

$$\psi(1) = \mathrm{Ev}_\lambda(1) = 1,$$

(1) tells us that ψ is also a state. Since the Gelfand transform takes a to the function $f(z) = z$, it follows that

$$|\psi(a)| = |\lambda| = r(a) = \|a\|$$

The last equality follows since a is self-adjoint.

This completes the proof. □

We are now ready to prove Theorem 18.1.

Proof. (Theorem 18.1) Let $D \subseteq A$ be a dense subset. For each $a \in D$, let $\varphi_a \in S(A)$ be such that $|\varphi_a(a^*a)| = \|a^*a\| = \|a\|^2$. Let $(\pi_a, \mathscr{H}_a, x_{\varphi_a})$ representation constructed in Proposition 17.2. Consider the direct sum representation:

$$\pi := \bigoplus_{a\in D} \pi_a : A \to \mathscr{B}\left(\bigoplus_{a\in D} \mathscr{H}_a\right) := \mathscr{B}(\mathscr{H}).$$

Assume $a \neq 0 \in A$ such that $\pi(a) = 0$. Then

$$\|\pi_a(a)x_{\varphi_a}\|^2 = \langle \pi_a(a^*a)x_{\varphi_a}, x_{\varphi_a}\rangle_{\varphi_a} = \varphi_a(a^*a) = \|a^*a\| \neq 0$$

This shows that π is non-zero, a contradiction. Hence, π is injective. It is clear that π is a $*$-homomorphism since each π_a is a faithful $*$-homomorphism. □

Remark 18.3. *With a bit more work, one can show that each state constructed in Lemma 18.2 can be taken to be a pure state. Hence, π in the proof of Theorem 18.1 can be taken to be a direct sum of irreducible representations of A.*

[16] We know this exists since $\sigma_A(a)$ is a non-empty, closed subset.

Part 3. C^*-Algebras II: Examples & Constructions

19. Universal C^*-Algebras

20. Group C^*-Algebras

21. Crossed Product C^*-Algebras

22. Tensor Products of C^*-Algebras

23. Colimits of C^*-Algebras