Please read the following instructions carefully:

- There are **nine problems** in this exam.
- There is **one bonus** problem.
- Solve the **bonus** problem only at the end.
- You have **100 minutes** to complete the exam
- The point distribution is given in the table below.
- Please write each solution on a separate page.
- You **must have your camera on** during the exam.
- This is a **closed book, closed notes exam**. You must not consult any resource while attempting the exam.
- Upload your work to Gradescope.
- Submitting the exam implies you abide by the honor pledge stated below:

I pledge on my honor that I have not given or received any unauthorized assistance on this quiz/examination

- **Here are some formulas/identities you may find useful:**

$$1 + \cos(x) = 2\cos^2(x/2)$$

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	0	90

1. (10 points) A solid is generated by rotating the region above the x-axis, to the right of the y-axis and between the graphs of the functions the graph of the function $y = x^3$ and $y = x^{\frac{1}{3}}$ around the y-axis. Compute the volume of the solid.

2. (10 points) The parametric equation of a curve is given by the equations,

$$x(t) = \cos(t) \quad y(t) = t + \sin(t), \quad 0 \le t \le 2\pi.$$

Find the arc length of the the curve.

3. (10 points) Compute the following limits:

$$\lim_{x\to\infty} \frac{\ln(x)}{\ln(\ln(x))} \qquad\qquad \lim_{x\to 0} \frac{e^x}{1 - x - x^2}$$

4. Compute the following integrals:
   (a) (5 points)

$$\int \sin(\ln(x))\, dx$$

   (b) (5 points)

$$\int \sin^2(t)\cos^5(t)\, dt$$

5. Determine whether the following improper integrals converge:
   (a) (5 points)

$$\int_1^\infty \frac{e^x}{x^{\frac{1}{2}}}\, dx$$

   (b) (5 points)

$$\int_0^\infty \frac{e^{-x}}{1 + x^2}\, dx$$

6. (10 points) Write an approximation using the trapezoidal rule for the integral,

$$\int_0^1 x^{10}\sin(x)\, dx,$$

using $n = 5$ sub-intervals. **Do NOT find the final numerical answer.**

7. (10 points) Determine the radius and interval of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{nx^n}{2^n(n+1)}$$

8. (10 points) Find the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(x-1)^n}{4 \cdot 9 \cdot 14 \cdots (5n-1)}$$

9. (10 points) Express the following indefinite integral as a power series:

$$\int \frac{\sin(x)}{x}\, dx.$$

10. (5 points (bonus)) One cute application of the theory of infinite series is in the area of mathematics called **Fourier/Harmonic analysis**. Joseph Fourier asked the following question while solving the heat equation:

Can a periodic function $f(x)$ on $[-\pi, \pi]$ be expressed as $f(x) = \dfrac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(nx) \ + \ b_n \sin(nx)$?

Note: the similarity with usual Taylor series in which each term is a polynomial of the form x^n. This is an extension of the same idea.

(a) Assuming a function can be expressed as a Fourier series, show that the coefficients a_0, a_n and b_n *have to be* given by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx.$$

Note: This argument is in spirit exactly like the argument we presented for the Taylor series case, where we argued the coefficients of the Taylor series had to be $f^n(0)/n!$.

(b) Show that the Fourier series associated to the function $f(x) = |x|$ on $[-\pi, \pi]$ is given by the formula:

$$f(x) \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos(nx).$$

(c) Using the above Fourier series (and assuming its convergence to f), show that,

$$\sum_{n\geq 1,\text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Note: Woot! Now not only do we know the p-series with $p = 2$ is convergent, but we know a value for the sum!

(d) **For a freebie bonus point or so:** write down any interesting thoughts/questions you have about how to solve the preceding parts? Any thoughts on how to solve (a), (b)? Any thoughts on the comparison with infinite series? Or write down any interesting comments, feedback etc. about the bonus questions throughout the semester, or anything about the course. Anything really that comes to mind!

Note: I don't expect anyone to solve this problem, especially on an exam. This has been presented simply for your edification, and to convince you of the myriad of applications of calculus. In particular, Fourier/Harmonic analysis has applications in probability theory, signal processing, quantum mechanics, electromagnetism, theoretical computer science etc...

Thanks for being part of the class this summer. Good luck with everything!